Worksheet 11/04. Math 110, Fall 2015. Some solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Normal and Self-Adjoint Operators, Spectral Theorem

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$. If an inner product is not specified then it will be assumed to be the 'obvious' one.

1.

a) Give an example of an operator $T \in L(\mathbb{C}^2)$ that is not a normal operator. Explain carefully why you know it is not a normal operator.

b) Give an example of a diagonalisable operator $T \in L(\mathbb{C}^2)$ that is not normal. Justify your chosen example carefully.

c) Give an example of an operator $T \in L(\mathbb{C}^3)$ that is normal but not self-adjoint.

d) Give an example of an operator $T \in L(\mathbb{R}^2)$ that is diagonalisable but not self-adjoint.

e) Verify that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$; $v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$ is normal. Explain why it's not self-adjoint.

Solution:

a) Any non-diagonalisable operator will do. For example,

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \ ; \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x.$$

b) To define a diagonalisable operator that is not normal it suffices to choose a basis of $\mathbb{C}^2$ that is not orthonormal with respect to the standard Hermitian inner product. For example, the basis $B = (v_1, v_2)$, where $v_1 = e_1, v_2 = e_1 + e_2$ is not orthogonal. Then, we can define $T \in L(\mathbb{C}^2)$ to be the unique operator such that $T(v_1) = v_1$, $T(v_2) = -v_2$. This is obviously diagonalisable, but not normal: the matrix of T with respect to the standard orthonormal basis is

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

As $AA^* \neq A^*A$ the operator can't be normal.

c) We can choose any orthonormal basis of $\mathbb{C}^3$ (with respect to the standard inner product) and use this to define a normal operator. So, take the standard basis (e_1, e_2, e_3). Then, to ensure that T is not self-adjoint, we define an operator with at least one nonreal eigenvalue: for example we can take the unique operator $T \in L(\mathbb{C}^3)$ such that $T(e_1) = e_1$, $T(e_2) = -e_2$ and $T(e_3) = ie_3$.

d) We choose an basis of $\mathbb{R}^2$ that is not orthonormal (with respect to the standard inner product). For example, choose that basis $(e_1, e_1 + e_2)$. Then, we define the diagonalisable operator $T \in L(\mathbb{R}^2)$ such that $T(e_1) = e_1$ and $T(e_1 + e_2) = -e_1 - e_2$.

e) The operator is not self-adjoint because it does not admit any real eigenvalues; hence, it is not diagonalisable, which violates the Real Spectral Theorem.

2. (*Longer?*) Repeat 1a)-d), replacing '$T \in L(\mathbb{C}^k)$' with '$T \in L(P_2(\mathbb{R})$', where $P_2(\mathbb{R})$ admits the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

3. Let $(\mathbb{R}^2, \langle , \rangle)$ be the inner product space, with

$$\langle \underline{x}, \underline{y}\rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2,\ \underline{x}, \underline{y} \in \mathbb{R}^2.$$

a) Define a self-adjoint operator T on the inner product space $(\mathbb{R}^2, \langle , \rangle)$ that has eigenvalues $\sqrt{2}, 1$.

b) Is the linear operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \ ;\ \underline{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x},$$

a self-adjoint operator on the inner product space $(\mathbb{R}^2, \langle , \rangle)$?

Solution:

a) We need to find an orthonormal basis of $\mathbb{R}^2$ with respect to the above inner product. One way is to start from **any** basis and use Gram-Schmidt. Or, you can check that $B = (e_2, e_1 + e_2)$ is orthonormal. Then, define $T \in L(\mathbb{R}^2)$ to be the unique operator such that $T(e_2) = \sqrt{2}e_2$, $T(e_1 + e_2) = e_1 + e_2$. Since T is diagonalisable and its eigenvectors are orthonormal, it must be self-adjoint by the Real Spectral Theorem.

b) We determine the basis of T with respect to the orthonormal basis given above: we find

$$A = [T]_B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \implies A^t = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Since $A \neq A^t$ this operator is not self-adjoint.

4. Let $(V, \langle , \rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Prove that T is self-adjoint if and only if all of the eigenvalues of T are real.

Solution: If T is self-adjoint then we can choose an orthonormal basis B for V consisting of eigenvectors of T. Then, the matrix of T with respect to this basis, let's call it A, is diagonal with $\lambda_1, ...\lambda_n$ appearing on the diagonal. Moreover, we must have $A = A^*$ because T is self-adjoint, and the matrix of T^* with resepct to B is A^*. Hence, we require that $\overline{\lambda_i} = \lambda_i$, for each i. The result follows.

Conversely, if all eigenvalues of a normal operator are real then we can use the spectral theorem again to find an orthonormal basis B of V consisting of eigenvectors of T. The matrix of T with respect to this basis, let's call it A, is diagonal and has only real entries appearing. Hence, the matrix of T^* is equal to $A^* = A$. Thus, the matrix of T and T^* (with respect to the same basis!) are equal so that $T = T^*$.

5. Let $(V, \langle , \rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Suppose that $T^{10} = T^8$. Prove that T is self-adjoint and that $T^3 = T$.

Solution: The relation $T^{10} = T^8$ implies that any eigenvalue λ of T must satisfy $\lambda^{10} = \lambda^8 \implies \lambda^8(\lambda^2 - 1) = 0$. Hence, the allowed eigenvalues are $\lambda = -1, 0, 1$. Since T is normal it is diaogonalisable so that there exists a basis of eigenvectors of T, $(v_1, \dots, v_n)$. Then, we have $T(v_i) = \lambda v_i$, where $\lambda \in \{-1, 0, 1\}$. Thus, regardless of the value of λ, $T^3(v_i) = \lambda^3 v_i = \lambda v_i = T(v_i)$. Therefore, T^3 and T agree on a basis so must be the same operator ie $T^3 = T$.

6. Let $(V, \langle,\rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Prove or give a counterexample: if $T^5 = 0 \in L(V)$ then $T = 0 \in L(V)$.

Solution: The allowed eigenvalue of T must satisfy $\lambda^5 = 0$. Hence, the only eigenvalue of T is $\lambda = 0$. Therefore, because T is normal, it is diagonalisable and the only diagonalisable operator with the single eigenvalue 0 is the zero operator.

7. Let $(V, \langle,\rangle)$ be an inner product space (over F), $T \in L(V)$ a normal operator.

a) Let $F = \mathbb{C}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

b) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

c) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^5 = T$.

Solution: First, we choose an orthonormal basis of V consisting of eigenvectors of T, $B = (v_1, \dots, v_n)$. Write the corresponding (not necessarily distinct!) eigenvalues as $\lambda_1, \dots, \lambda_n$.

a) Choose $\mu_i \in \mathbb{C}$ such that $\mu_i^4 = \lambda_i$. Then, we define $S \in L(V)$ to be the unique operator such that $S(v_i) = \mu_i v_i$. Thus, we see that $S^4(v_i) = \mu_i^4 v_i = \lambda_i v_i = T(v_i)$ and therefore $S^4 = T$ (because these operator agree on a basis).

8. Let $(V, \langle,\rangle)$ be an inner product space (over $\mathbb{C}$), $T \in L(V)$ an operator (not necessarily normal/self-adjoint!). Prove or give a counterexample:

a) if T admits exactly two eigenvalues 1 and $-i$ and $E(1, T) \subset E(-1, T)^\perp$ then T is normal.

b) if T admits exactly two eigenvalues 1 and -1 and $E(1, T) = E(-1, T)^\perp$ then T is self-adjoint.

9*. (*Harder*) Let $(V, \langle,\rangle)$ be a complex inner product space, $S, T \in L(V)$ normal operators. Prove: there exists a basis $B \subset V$ consisting of eigenvectors of both S and T if and only if $ST = TS$.

10*. (*Harder*) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with complex entries. Say that A is **normal** if $AA^* = A^*A$, where $A^* = \overline{A}^t$ is the conjugate transpose. Give conditions on a, b, c, d so that A is normal and admits two distinct eigenvalues. What if you want A to be normal and have exactly one eigenvalue?