Worksheet 11/04. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Normal and Self-Adjoint Operators, Spectral Theorem

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$. If an inner product is not specified then it will be assumed to be the 'obvious' one.

1.

a) Give an example of an operator $T \in L(\mathbb{C}^2)$ that is not a normal operator. Explain carefully why you know it is not a normal operator.

b) Give an example of a diagonalisable operator $T \in L(\mathbb{C}^2)$ that is not normal. Justify your chosen example carefully.

c) Give an example of an operator $T \in L(\mathbb{C}^3)$ that is normal but not self-adjoint.

d) Give an example of an operator $T \in L(\mathbb{R}^2)$ that is diagonalisable but not self-adjoint.

e) Verify that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2 \ ; \ v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$ is normal. Explain why it's not self-adjoint.

2. (*Longer?*) Repeat 1a)-d), replacing '$T \in L(\mathbb{C}^k)$' with '$T \in L(P_2(\mathbb{R})$', where $P_2(\mathbb{R})$ admits the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

3. Let $(\mathbb{R}^2, \langle , \rangle)$ be the inner product space, with

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2, \ \underline{x}, \underline{y} \in \mathbb{R}^2.$$

a) Define a self-adjoint operator T on the inner product space $(\mathbb{R}^2, \langle , \rangle)$ that has eigenvalues $\sqrt{2}, 1$.

b) Is the linear operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \ ; \ \underline{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x},$$

a self-adjoint operator on the inner product space $(\mathbb{R}^2, \langle , \rangle)$?

4. Let $(V, \langle , \rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Prove that T is self-adjoint if and only if all of the eigenvalues of T are real.

5. Let $(V, \langle , \rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Suppose that $T^{10} = T^8$. Prove that T is self-adjoint and that $T^3 = T$.

6. Let $(V, \langle , \rangle)$ be a complex inner product space, $T \in L(V)$ a normal operator. Prove or give a counterexample: if $T^5 = 0 \in L(V)$ then $T = 0 \in L(V)$.

7. Let $(V, \langle , \rangle)$ be an inner product space (over F), $T \in L(V)$ a normal operator.

a) Let $F = \mathbb{C}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

b) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^4 = T$.

c) Let $F = \mathbb{R}$. Prove or give a counterexample: there exists an operator $S \in L(V)$ such that $S^5 = T$.

8. Let $(V, \langle , \rangle)$ be an inner product space (over $\mathbb{C}$), $T \in L(V)$ an operator (not necessarily normal/self-adjoint!). Prove or give a counterexample:

a) if T admits exactly two eigenvalues 1 and $-i$ and $E(1, T) \subset E(-1, T)^{\perp}$ then T is normal.

b) if T admits exactly two eigenvalues 1 and -1 and $E(1, T) = E(-1, T)^{\perp}$ then T is self-adjoint.

9*. (*Harder*) Let $(V, \langle , \rangle)$ be a complex inner product space, $S, T \in L(V)$ normal operators. Prove: there exists a basis $B \subset V$ consisting of eigenvectors of both S and T if and only if $ST = TS$.

10*. (*Harder*) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with complex entries. Say that A is **normal** if $AA^* = A^*A$, where $A^* = \overline{A}^t$ is the conjugate transpose. Give conditions on a, b, c, d so that A is normal and admits two distinct eigenvalues. What if you want A to be normal and have exactly one eigenvalue?