Worksheet 10/21. Math 110, Fall 2015. Some solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Diagonalisability

1. Recall: if $p = a_0 + ... + a_n x^n \in P(\mathbb{C})$ and $p(T) = 0 \in L(V)$, then any eigenvalue of T must be a root of p, ie, if $\lambda \in \mathbb{C}$ is an eigenvalue of T then $p(\lambda) = 0$. (*This was a homework problem*)

i) Consider the linear map

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \ ; \ v \mapsto \begin{bmatrix} -1 & -1 & 1 \\ 2 & -4 & 2 \\ 0 & 0 & 3 \end{bmatrix} v.$$

a) T satisfies $p(T) = 0$, where $p = x(x-2)(x-3)^2(x-1)$. What are the allowed eigenvalues of T?

b) What are the eigenvalues of T?

c) What are the dimensions of the eigenspaces for T?

d) Is T diagonalisable?

Solution: Write $T(v) = Av$, where A is matrix above.

a) $\lambda = 0, 2, 3, 1$

b) $\lambda = 2, 3$ - to see this, you check for each λ in (a) whether $\text{nul}(T - \lambda) \neq \{0\}$. Those λ for which this is true are the eigenvalues of T.

c) $\dim E(\lambda, T) = \dim \text{nul}(T - \lambda \text{id})$. Thus, we row-reduce the matrix $A - \lambda I_3$ and count the number of non-pivot columns. We see that $\dim E(2, T) = 1$, $\dim E(3, T) = 2$.

d) Since $\dim E(2, T) + \dim E(3, T) = 3 = \dim \mathbb{R}^3$, we have that T is diagonalisable.

ii) Consider the linear map

$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \ ; \ p \mapsto p + p' + p''.$$

a) Write down the linear maps $A = S - \text{id}$, $B = (S - \text{id})^2$ and $C = (S - \text{id})^3$; that is, what are the outputs $A(p), B(p), C(p)$, given any $p \in P_2(\mathbb{R})$?

b) Using (a), show that S can have at most one eigenvalue. Prove that S admits an eigenvector for this eigenvalue.

c) What is the dimension of the eigenspace?

d) Is S diagonalisable?

Solution:

a) We have $A(p) = p' + p''$. Then,

$$B(p) = (S - \mathrm{id})(A(p)) = (S - \mathrm{id})(p' + p'')$$

$$= (p' + p'') + (p'' + p''') + (p''' + p'''') - (p' + p'') = (p'' + p''') + (p''' + p'''') = p'',$$

since $p \in P_2$. Finally, $C(p) = (S - \mathrm{id})(B(p)) = (S - \mathrm{id})(p'') = p'' + p''' + p'''' - p'' = 0$.

b) Since S satisfies the polynomial relation $(x-1)^3 = 0$, we see that the only eigenvalue can be $\lambda = 1$. Furthermore, $S(1) = 1$, so that $p = 1 \in P_2$ is an eigenvector wih eigenvalue $\lambda = 1 \in \mathbb{R}$.

c) If p is an eigenvector with eigenvalue $\lambda = 1$ then we must have $S(p) = p$. Then, we require $p = S(p) = p + p' + p'' \implies p' + p'' = 0 \in P_2$. If $p = a + bx + cx^2$, this means that we require $0 = (b + 2cx) + 2c = (b + 2c) + 2cx \implies c = b = 0$. Hence, $p \in \mathrm{span}(1)$. Thus, $\dim E(1, S) = 1$.

d) No S is not diagonalisable since $\dim E(1, S) = 1 < 3$.

2. Give an example of an operator $T : F^2 \to F^2$; $v \mapsto Av$, for a 2×2 matrix A, that is not diagonalisable when $F = \mathbb{R}$, but is diagonalisable when $F = \mathbb{C}$.

Solution: The operator

$$T : F^2 \to F^2 \; ; \; v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v,$$

is not diagonalisable when $F = \mathbb{R}$ (it's the rotation by $\pi/2$ counter-clockwise map, hence there are no 'fixed lines'), but is diagonalisable when $F = \mathbb{C}$ - it has two distinct eigenvalues $\lambda = \pm i$.

3. Let V be finite dimensional complex vector space, $S, T \in L(V)$ such that $ST = TS$. Prove: any eigenspace $E(\lambda, T)$ of T is S-invariant.

Solution: Let $v \in E(\lambda, T)$, so that $T(v) = \lambda v$. Then, $T(S(v)) = S(T(v)) = S(\lambda v) = \lambda S(v)$, so that $S(v) \in E(\lambda, T)$.

4. Let V be a finite dimensional vector space (over F), and suppose that $T \in L(V)$ is diagonalisable, and $S \in L(V)$ satisfies $ST = TS \in L(V)$. Prove or give a counterexample:

a) If $F = \mathbb{R}$ then S is diagonalisable.

b) If $F = \mathbb{C}$ then S is diagonalisable.

c) If $F = \mathbb{C}$ and T has $n = \dim V$ distinct eigenvalues, then S is diagonalisable.

Solution:

a) Let $T = \mathrm{id} \in L(\mathbb{R}^2)$ and S be the rotation by $\pi/2$ map defined above. Then, S is not diagonalisable but $TS = ST$.

b) Let $T = \mathrm{id} \in L(\mathbb{C}^2)$ and $S \in L(\mathbb{C}^2)$ be the map $S(ae_1 + be_2) = (a + b)e_1 + be_2$. Then, $ST = TS$ but S is not diagonalisable: the matrix of S with respect to the standard basis is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, so that there is only one eigenvalue $\lambda = 1$ (recall: eigenvalues can be read off the diagonal of any upper triangular matrix representation of S). Moreover, $E(1, S) = \mathrm{span}(e_1)$, so that $\dim E(1, S) = 1 < 2$.

5. Suppose that $T \in L(\mathbb{C}^4)$ has eigenvalues $1, 2, 3$. Prove: if T is not invertible then $T + \text{id}$ is invertible.

Solution: If T is not invertible then $\text{nul}(T) \neq \{0\}$, so that 0 is an eigenvalue of T. Hence, T has distinct eigenvalues $0, 1, 2, 3$. Any operator on $\mathbb{C}^4$ has at most 4 distinct eigenvalues, so that there are no further eigenvalues. In particular, $\text{nul}(T - (-1)\text{id}) = \{0\}$ (otherwise -1 is an eigenvalue). Hence, $T + \text{id}$ is injective and therefore an isomorphism.