Worksheet 10/21. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Diagonalisability

1. Recall: if $p = a_0 + ... + a_n x^n \in P(\mathbb{C})$ and $p(T) = 0 \in L(V)$, then any eigenvalue of T must be a root of p, ie, if $\lambda \in \mathbb{C}$ is an eigenvalue of T then $p(\lambda) = 0$. (*This was a homework problem*)

i) Consider the linear map

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \; ; \; v \mapsto \begin{bmatrix} -1 & -1 & 1 \\ 2 & -4 & 2 \\ 0 & 0 & 3 \end{bmatrix} v.$$

a) T satisfies $p(T) = 0$, where $p = x(x-2)(x-3)^2(x-1)$. What are the allowed eigenvalues of T?

b) What are the eigenvalues of T?

c) What are the dimensions of the eigenspaces for T?

d) Is T diagonalisable?

ii) Consider the linear map

$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \; ; \; p \mapsto p + p' + p''.$$

a) Write down the linear maps $A = S - \mathrm{id}$, $B = (S - \mathrm{id})^2$ and $C = (S - \mathrm{id})^3$; that is, what are the outputs $A(p), B(p), C(p)$, given any $p \in P_2(\mathbb{R})$?

b) Using (a), show that S can have at most one eigenvalue. Prove that S admits an eigenvector for this eigenvalue.

c) What is the dimension of the eigenspace?

d) Is S diagonalisable?

2. Give an example of an operator $T : F^2 \to F^2 \; ; \; v \mapsto Av$, for a 2×2 matrix A, that is not diagonalisable when $F = \mathbb{R}$, but is diagonalisable when $F = \mathbb{C}$.

3. Let V be finite dimensional complex vector space, $S, T \in L(V)$ such that $ST = TS$. Prove: any eigenspace $E(\lambda, T)$ of T is S-invariant.

4. Let V be a finite dimensional vector space (over F), and suppose that $T \in L(V)$ is diagonalisable, and $S \in L(V)$ satisfies $ST = TS \in L(V)$. Prove or give a counterexample:

a) If $F = \mathbb{R}$ then S is diagonalisable.

b) If $F = \mathbb{C}$ then S is diagonalisable.

c) If $F = \mathbb{C}$ and T has $n = \dim V$ distinct eigenvalues, then S is diagonalisable.

5. Suppose that $T \in L(\mathbb{C}^4)$ has eigenvalues $1, 2, 3$. Prove: if T is not invertible then $T + \mathrm{id}$ is invertible.