Worksheet 10/14. Math 110, Fall 2015. Some solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Eigenstuff/Invariant subspaces

1. Consider the operator

$$T : F^2 \to F^2 \; ; \; v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v.$$

a) Find the eigenvalues/eigenvectors for T when $F = \mathbb{C}$.

b) Are there any eigenvectors/eigenvalues when $F = \mathbb{R}$?

Solution: An eigenvector is a nonzero vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $T(v) = \lambda v$, for some λ. So, we see that v_1, v_2 must satisfy $v_1 = \lambda v_2$, $-v_2 = \lambda v_1$. Hence, we combine these equations to obtain $v_2 = -\lambda^2 v_2$ and $v_1 = -\lambda^2 v_1$. Since at least one of v_1, v_2 are nonzero, we find that $\lambda^2 = -1$, and the allowed eigenvalues are $\lambda = \pm i$. The above equations show that $v_1 \neq 0 \Leftrightarrow v_2 \neq 0$, so we find the eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ for $\lambda = i$, and the eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ for $\lambda = -i$. There no real solutions to $\lambda^2 = -1$ so that there are no real eigenvalues; hence, no eigenvectors.

2. Let $(v_1, \dots, v_k)$ be a list of eigenvectors for $T : V \to V$. Is $U = \mathrm{span}(v_1, \dots, v_k)$ T-invariant? Prove or provide a counterexample.

Solution: Let $v = a_1 v_1 + \dots + a_k v_k \in \mathrm{span}(v_1, \dots, v_k)$. Then, since $T(v_j) = \lambda_j v_j$ (here $\lambda_1, \dots, \lambda_k$ are the eigenvalues of $v_1, \dots, v_k$)), then we have

$$T(v) = a_1 T(v_1) + \dots + a_k T(v_k) = a_1 \lambda_1 v_1 + \dots + a_k \lambda_k v_k \in \mathrm{span}(v_1, \dots, v_k).$$

Hence, $\mathrm{span}(v_1, \dots, v_k)$ is T-invariant.

3. **True/False**: Let V be a vector space, $T, S \in L(V)$.

a) If $v \in V$ is an eigenvector for T and S, then it is an eigenvector for $T + S$.

b) If λ is an eigenvalue for T and S, then it is an eigenvalue for $T + S$.

c) If v, u are eigenvectors for T then $v + u$ is an eigenvector for T.

Solution:

a) True: if $T(v) = av$, $S(v) = bv$, for some a, b, then $(T + S)(v) = T(v) + S(v) = av + bv = (a + b)v$.

b) False: let $S = T = \mathrm{id}_{\mathbb{R}^2}$, both having eigenvalue 1. Then, $S + T = 2\mathrm{id}_{\mathbb{R}^2}$.

c) False: consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \; x \mapsto Ax$, where A is diagonal with $1, 2$ on the diagonal.

4. Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \ ; \ v \mapsto \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} v$$

Find a 1-dimensional T-invariant subspace U and a 2-dimensional subspace T-invariant subspace W.

Solution: It is straightforward to see (by looking at the matrix defining T) that $U = \text{span}(e_1, e_2)$ is T-invariant: indeed $T(e_1) = 2e_1 + e_2 \in U$ and $T(e_2) = e_1 + 2e_2 \in U$. Since U is T-invariant, we can consider the <u>new</u> operator $S \in L(U)$, where $S : U \to U \ ; \ u \mapsto T(u)$ (ie S is the restriction of T to U). So, if we can find an S-invariant line $L \subset U$, then L will also be T-invariant: if $x \in L$ then $T(x) = S(x) \in L$. Thus, we require $L = \text{span}(v)$, where $v = ae_1 + be_2 \neq 0$, and so that $S(v) = \lambda v$, for some λ. As $S(v) = aS(e_1) + bS(e_2) = (2a + b)e_1 + (a + 2b)e_2$, we must have

$$(2a + b) = \lambda a, \quad (a + 2b) = \lambda b, \quad \text{for some } \lambda.$$

Since we require that $v \neq 0 \in U$, we must have that either $a \neq 0$ or $b \neq 0$ - the equations show that $a = 0$ if and only if $b = 0$. Hence, we can assume, without loss of generality, that $a \neq 0$. Hence, using $b = (\lambda - 2)a$ (from the first equation), the second equation gives $a + 2(\lambda - 2)a = \lambda(\lambda - 2)a \implies a(\lambda^2 - 4\lambda + 3) = 0 \implies (\lambda^2 - 4\lambda + 3) = 0$, since $a \neq 0$. So, the allowed λ are $\lambda = 1, 3$. When $\lambda = 1$ we solve the equations above to obtain $a = -b$, so that $v = e_1 - e_2$ is an eigenvector of S (and also of T!) with eigenvalue $\lambda = 1$. Thus, take $L = \text{span}(v)$.

5. Consider the operator

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \ ; \ v \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} v.$$

a) Find a 1-dim T-invariant subspace $U \subset \mathbb{C}^3$, where $U \neq \{0\}, \mathbb{C}^3$.

b) Find a 2-dim T-invariant subspace $V \subset \mathbb{C}^3$, where $V \neq \{0\}, \mathbb{C}^3, U$.

Solution:

a) The second column of the matrix implies that $\text{span}(e_2)$ is T-invariant.

b) It can be seen that $\text{span}(e_1, e_3)$ is T-invariant.

6. Let $T \in L(V)$ and suppose that $U, W \subset V$ are T-invariant subspaces. Is $U \cap W$ T-invariant? Prove or give a counterexample. What about $U + W$? Prove or give a counterexample.

Solution: $U \cap W$ is T-invariant. Let $x \in U \cap W$. Then, $T(x) \in U$ (since U is T-invariant) and $T(x) \in W$ (since W is T-invariant). Hence, $T(x) \in U \cap W$. Also, $U + W$ is T-invariant: if $x = u + w$ then $T(x) = T(u) + T(w) \in U + W$.

7. Consider the linear operator

$$T : P(\mathbb{R}) \to P(\mathbb{R}) \ ; \ p(x) \mapsto (xp(x))'$$

Show that T has an infinite number of eigenvalues.

Solution: Observe that $T(x^n) = (n+1)x^n$, for every n. Hence, every positive integer is an eigenvalue.

8. Recall that an operator $T \in L(V)$ is diagonalisable if there exists a basis B of V consisting of eigenvectors of T. Assume that $F = \mathbb{C}$. Give examples of the following types of operators or explain why such an operator can't exist: (look for $T \in L(F^2)$, defined by a 2×2 matrix A)

i) diagonalisable, invertible,

ii) diagonalisable, not invertible,

iii) not diagonalisable, invertible,

iv) not diagonalisable, not invertible.

Do you think there is a relationship between invertibility and diagonalisability?

9. (*Harder*) Consider the following operators

$$S_1 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}, \quad S_3 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x}$$

i) Verify that $S^2 = \mathrm{id}_{\mathbb{C}^2}$ and $R^2 = \mathrm{id}_{\mathbb{C}^2}$ and that $S_1 S_3 = S_3 S_1$.

ii) What are the allowed eigenvalues of S and R? Show that both S and R must admit two distinct eigenvalues $1, -1$.

iii) Suppose that $S_1(w) = -w$. Show that $S_3(w)$ is also an eigenvector of S_1 with eigenvalue -1. Show that $\dim \mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^2}) = 1$. Show that $S_3(w) = w$.

iv) Determine a basis B of $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ (Hint: $\dim \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^3}) = 3$)

v) Explain why $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ is S_3-invariant.

vi) Consider the restriction operator

$$S_3' : \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \to \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \; ; \; \underline{x} \mapsto S_3(\underline{x}).$$

Find the matrix $[S_3']_B$ of S_3' with respect to B. (Hint: it should be 3×3)

vii) Find two distinct eigenvalues and three linearly independent eigenvectors (v_1, v_2, v_3) of S_3'.

viii) Show that $C = (w, v_1, v_2, v_3)$ is linearly independent, and determine $[S_1]_C, [S_3]_C$. What do you notice?