Worksheet 10/14. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Eigenstuff/Invariant subspaces

1. Consider the operator

$$T : F^2 \to F^2 \; ; \; v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v.$$

a) Find the eigenvalues/eigenvectors for T when $F = \mathbb{C}$.

b) Are there any eigenvectors/eigenvalues when $F = \mathbb{R}$?

2. Let $(v_1, \dots, v_k)$ be a list of eigenvectors for $T : V \to V$. Is $U = \text{span}(v_1, \dots, v_k)$ T-invariant? Prove or provide a counterexample.

3. **True/False**: Let V be a vector space, $T, S \in L(V)$.

a) If $v \in V$ is an eigenvector for T and S, then it is an eigenvector for $T + S$.

b) If λ is an eigenvalue for T and S, then it is an eigenvalue for $T + S$.

c) If v, u are eigenvectors for T then $v + u$ is an eigenvector for T.

4. Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \; ; \; v \mapsto \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} v$$

Find a 1-dimensional T-invariant subspace U and a 2-dimensional subspace T-invariant subspace W.

5. Consider the operator

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \; ; \; v \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix} v.$$

a) Find a 1-dim T-invariant subspace $U \subset \mathbb{C}^3$, where $U \neq \{0\}, \mathbb{C}^3$.

b) Find a 2-dim T-invariant subspace $V \subset \mathbb{C}^3$, where $V \neq \{0\}, \mathbb{C}^3, U$.

6. Let $T \in L(V)$ and suppose that $U, W \subset V$ are T-invariant subspaces. Is $U \cap W$ T-invariant? Prove or give a counterexample. What about $U + W$? Prove or give a counterexample.

7. Consider the linear operator

$$T : P(\mathbb{R}) \to P(\mathbb{R}) \; ; \; p(x) \mapsto (xp(x))'$$

Show that T has an infinite number of eigenvalues.

8. Recall that an operator $T \in L(V)$ is diagonalisable if there exists a basis B of V consisting of eigenvectors of T. Assume that $F = \mathbb{C}$. Give examples of the following types of operators or explain why such an operator can't exist: (look for $T \in L(F^2)$, defined by a 2×2 matrix A)

i) diagonalisable, invertible,

ii) diagonalisable, not invertible,

iii) not diagonalisable, invertible,

iv) not diagonalisable, not invertible.

Do you think there is a relationship between invertibility and diagonalisability?

9. (*Harder*) Consider the following operators

$$S_1 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}, \quad S_3 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x}$$

i) Verify that $S^2 = \mathrm{id}_{\mathbb{C}^2}$ and $R^2 = \mathrm{id}_{\mathbb{C}^2}$ and that $S_1 S_3 = S_3 S_1$.

ii) What are the allowed eigenvalues of S and R? Show that both S and R must admit two distinct eigenvalues $1, -1$.

iii) Suppose that $S_1(w) = -w$. Show that $S_3(w)$ is also an eigenvector of S_1 with eigenvalue -1. Show that $\dim \mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^2}) = 1$. Show that $S_3(w) = w$.

iv) Determine a basis B of $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ (Hint: $\dim \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^3}) = 3$)

v) Explain why $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ is S_3-invariant.

vi) Consider the restriction operator

$$S_3' : \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \to \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \; ; \; \underline{x} \mapsto S_3(\underline{x}).$$

Find the matrix $[S_3']_B$ of S_3' with respect to B. (Hint: it should be 3×3)

vii) Find two distinct eigenvalues and three linearly independent eigenvectors (v_1, v_2, v_3) of S_3'.

viii) Show that $C = (w, v_1, v_2, v_3)$ is linearly independent, and determine $[S_1]_C, [S_3]_C$. What do you notice?