Worksheet 10/7. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Dual things; annhilators

1. Consider the linear map

$$T : F^3 \to F^4 \; ; \; v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 - 2v_3 \\ 0 \\ v_1 + v_2 + v_3 \\ 3v_2 + v_3 \end{bmatrix}.$$

i) Let $S_4 \subset F^4$ be the standard basis, and let $S_4' = (y_1, \dots, y_4)$ be the dual basis of S_4'; similarly, let $S_3 \subset F^3$ be the standard basis and let $S_3' = (x_1, x_2, x_3)$ be the dual basis of S_3. What is $T'(y_i)$, for each $i = 1, \dots, 4$? (Here $T' \in L((F^4)', (F^3)')$ is the dual map of T)

ii) Consider the linear functional

$$\alpha : F^4 \to F \; ; \; u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \mapsto -u_1 + 3u_3 + u_4.$$

Write α as a linear combination of S_4', ie, find scalars $a, b, c, d \in F$ such that

$$\alpha = ay_1 + by_2 + cy_3 + dy_4.$$

iii) Write $T'(\alpha) \in (F^3)'$ as a linear combination of S_3', ie, find scalars $p, q, r \in F$ such that

$$T'(\alpha) = px_1 + qx_2 + rx_3.$$

2. Do the same problems above (with appropriate adjustments!) for the linear map

$$T : F^3 \to F^2 \; ; \; v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mapsto \begin{bmatrix} v_1 + v_2 - v_3 \\ v_1 + 2v_2 + 3v_3 \end{bmatrix},$$

and the linear functional

$$\alpha : F^2 \to F \; ; \; v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto v_1 - v_2.$$

3. **FACT:** Elements of $(F^n)'$ may be thought of as the vector space $Mat_{1,n}(F)$ of $1 \times n$ matrices (with entries in F). Make this identification.

i) Given a vector $v \in F^n$ and $A \in Mat_{1,n}(F)$, verify that Av can be considered as a scalar.

ii) Let $B = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) \subset \mathbb{C}^2$, a basis of $\mathbb{C}^2$. What are the row-vectors (ie, $1 \times n$ matrices) $A_1, A_2 \in Mat_{1,n}(\mathbb{C})$ such that (A_1, A_2) is the dual basis of B? (*Hint: what equations must A_1 and A_2 satisfy?*)

iii) Write A_1, A_2 as linear combinations of elements of the dual basis of S_2 (the standard basis of $\mathbb{C}^2$). (You'll need to think which row vectors the elements of S_2' correspond to...)

iv) Consider the linear map

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; v \mapsto \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} v.$$

If we denote $S_2' = (x_1, x_2)$, then write $T'(x_1)$ and $T'(x_2)$ as linear combinations of S_2'. Can you find a matrix $A \in Mat_2(\mathbb{C})$ such that

$$T' : (\mathbb{C}^2)' \to (\mathbb{C}^2)' \; ; \; \alpha \mapsto \alpha A?$$

Here we are considering elements of $(\mathbb{C}^2)'$ as row-vectors. Is the matrix A related to the matrix defining T in any way?

4. Let

$$U = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in F^4 \;\middle|\; \begin{matrix} v_1 + v_4 = 0 \\ v_2 - v_3 = 0 \end{matrix} \right\} \subset F^4;$$

this is a subspace of F^4.

i) Determine a basis $B = (u_1, u_2)$ of U. Extend this basis to a basis C of F^4.

ii) Let $C' = (\alpha_1, \dots, \alpha_4)$ be the dual basis of C. Write α_i as a linear combination of the elements of $S_4' = (x_1, x_2, x_3, x_4)$ (the dual basis of the standard basis of F^4).

iii) Prove that (α_3, α_4) is a basis of U°, the annhilator of U. (*Hint: you don't need to show that these functionals span U°!*)

iv) Prove that $x_1 + x_4, x_2 - x_3 \in U^\circ$.

v) (*Harder*) Suppose that A is an $m \times n$ matrix with entries in F. Suppose that $U = \text{null}(A) \subset F^n$. Can you find a spanning list of U°? Can you find a basis of U°?