Worksheet 9/30. Math 110, Fall 2015. Some solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Quotients

1. Find a basis for the following quotients V/U, and prove that your list is indeed a basis.

a) $V = \mathbb{R}^2$, $U = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$,

b) $V = \mathbb{R}^3$, $U = \{x \mid x_1 + x_2 + x_3 = 0\}$,

c) $V = \mathbb{R}^3$, $U = \text{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$.

d) $V = \mathbb{R}^{2,2}$, $U = \{A \mid A = -A^t\}$, where

$$A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \implies A^t = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$$

Solution:

a) We have $\dim V/U = \dim V - \dim U = 1$. Hence, we need to only choose a nonzero vector in $v \in V/U$; $v = e_1 + U \in V/U$ suffices.

b) $\dim U = 2$ so that $\dim V/U = 1$. Again, we need only choose a nonzero $v + U \in V/U$ to obtain a basis. Any $v \in \mathbb{R}^3$ whose coordinates do not sum to zero will work, eg $e_1 + U \in V/U$.

c) We have $\dim V/U = 2$; hence we must choose two linear independent vectors $v_1, v_2 \in V/U$. If $v_1 = x_1 + U, v_2 = x_2 + U$ then we require that $v_1 \notin \text{span}(v_2)$, which means that there does not exist a scalar $a \in \mathbb{R}$ such that $v_1 = av_2$. This means there is no scalar such that $x_1 + U = ax_2 + U$. Thus, we need that $x_1 \notin \text{span}(x_2) + U \subset \mathbb{R}^3$. So, choose $x_2 \notin U$, say $x_2 = e_2$, and take $x_1 = e_1 \notin \text{span}(x_2) + U$. Then, $v_1 = e_1 + U, v_2 = e_2 + U$ is a basis of V/U.

d) The subspace U is simply $U = \text{span}\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)$. Thus, we need three linearly independent vectors. By similar reasoning as in the last problem, we can take $v_1 = e_{11} + U, v_2 = e_{12} + U, v_3 = e_{22} + U$, this is a linearly independent list: indeed if we have a linear relation

$$av_1 + bv_2 + cv_3 = 0 \in V/U \implies (ae_{11} + be_{12} + ce_{22}) + U = 0 + U$$

$$\implies ae_{11} + be_{12} + ce_{22} = \begin{bmatrix}a & b\\ 0 & c\end{bmatrix} \in U \implies a = b = c = 0$$

2. Prove or give a counterexample: Let $U \subset V$ be a subspace.

a) if $(v_1, \dots, v_n) \subset V$ is linearly independent then $(v_1 + U, \dots, v_n + U) \subset V/U$ is linearly independent.

b) if $(w_1 + U, \dots, w_n + U) \subset V/U$ is linearly independent then $(w_1, \dots, w_n) \subset V$ is linearly independent.

c) if $c_1(w_1 + U) + \dots + c_n(w_n + U) = 0_{V/U}$ is a linear relation, then $c_1 w_1 + \dots + c_n w_n = 0_V$.

d) if (v_1, v_2) are linearly independent and $v_1, v_2 \notin U$ then $(v_1 + U, v_2 + U)$ are linearly independent.

Solution:

a) False: $e_1, e_2 \in \mathbb{R}^2$ are linearly independent, but if $U = \text{span}(e_1 - e_2)$ then $e_1 + U = e_2 + U$.

b) This is a consequence of the following: if $T : V \to W$ is a linear map and $(T(v_1), \dots, T(v_n)) \subset W$ is linearly independent, then $(v_1, \dots, v_n)$ is linearly independent (prove it!). Then, apply this result to $T = \pi_U : V \to V/U$.

c) This is false: consider the example given in (a) above.

d) False: again, use (a).

3. Define an explicit isomorphism between $W = \mathbb{R}^2$ and V/U, where $V = \mathbb{R}^5$, $U = \{x \mid x_1 + x_3 + x_5 = 0, x_2 - x_4 = 0\}$. (ie, give linear maps $T : W \to V/U$ and $S : V/U \to W$ that are inverse functions)

Solution: Observe that $x = e_1 + U, y = e_2 + U$ are nonzero elements in V/U. Furthermore, they are linearly independent in V/U: suppose that we have a linear relation

$$ax + by = 0 \in V/U \implies (ae_1 + be_2) + U = 0 + U \implies ae_1 + be_2 \in U$$

Howwever, if either of a or b is nonzero then the vector $ae_1 + be_2$ does not satisfy the defining conditions of U. Hence, we must have $a = b = 0$. Thus, (x, y) is a basis of V/U. Now, define a linear map $T : \mathbb{R}^2 \to V/U$, such that $T(e_1) = x$, $T(e_2) = y$. Then, this gives rise to a linear map (Theorem 3.5) and it is injective: if $v = ae_1 + be_2 \in \text{nul}(T)$ then $ax + by = 0$, so that $a = b = 0$. Hence, $v = 0$ and $\text{nul}(T) = \{0\}$. As $\mathbb{R}^2$ and V/U have the same dimension, we find that T is an isomorphism.

4. Prove: let V be finite dimensional, $U \subset V$ a subspace. Then, there exists a subspace $W \subset V$ such that $V = U \oplus W$ and $\pi(W) = V/U$, where $\pi : V \to V/U$ is the quotient map.

Solution: Let $(u_1, \dots, u_k)$ be a basis of U and extend to a basis $(u_1, \dots, u_k, v_1, \dots, v_l)$ of V. Define $W = \text{span}(v_1, \dots, v_l)$. Then, $V = U \oplus W$ and, for any $\pi_U(v) = v + U \in V/U$ we have $v = u + w \in U + W$, and $v + U = \pi_U(u + w) = \pi_U(u) + \pi_U(w) = \pi_U(w)$. Hence, $\pi_U(W) = V/U$.

5. Consider the quotient space V/U from Exercise 1d). Let $W = \{A \in V \mid A = A^t\}$. Show that the linear map

$$\pi_{|W} : W \to V/U \; ; \; w \mapsto w + U$$

is an isomorphism.

Solution: Consider the sum $U + W \subset \mathbb{R}^{2,2}$. Note that $X \in W$ if and only if $X = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, so that $W = \text{span}(e_{11}, e_{22}, e_{12} + e_{21})$. Hence, $\dim W = 3$. Thus, we need only show that $\pi_{|W}$ is an injective map. If $X \in \text{nul}(\pi_{|W}) \subset W$ then $X + U = \pi_{|W}(X) = 0 + U \implies X \in U$. Thus, $X \in U \cap W$. Then, we must have that $X = X^t$ and $X = -X^t$. That is, $X^t = -X^t \implies X^t = 0 \implies X = 0$. The result follows.

6. (Harder) Let $V = P(\mathbb{R})$ and $U = \{p = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_k x^{2k} \mid k \in \mathbb{N}, a_1, \dots, a_k \in \mathbb{R}\}$. Prove that V/U is infinite dimensional.