Worksheet 9/30. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Quotients

1. Find a basis for the following quotients V/U, and prove that your list is indeed a basis.

a) $V = \mathbb{R}^2$, $U = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$,

b) $V = \mathbb{R}^3$, $U = \{x \mid x_1 + x_2 + x_3 = 0\}$,

c) $V = \mathbb{R}^3$, $U = \text{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$.

d) $V = \mathbb{R}^{2,2}$, $U = \{A \mid A = -A^t\}$, where

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix} \implies A^t = \begin{bmatrix}a & c\\b & d\end{bmatrix}$$

2. Prove or give a counterexample: Let $U \subset V$ be a subspace.

a) if $(v_1, \dots, v_n) \subset V$ is linearly independent then $(v_1 + U, \dots, v_n + U) \subset V/U$ is linearly independent.

b) if $(w_1 + U, \dots, w_n + U) \subset V/U$ is linearly independent then $(w_1, \dots, w_n) \subset V$ is linearly independent.

c) if $c_1(w_1 + U) + \dots + c_n(w_n + U) = 0_{V/U}$ is a linear relation, then $c_1w_1 + \dots + c_nw_n = 0_V$.

d) if (v_1, v_2) are linearly independent and $v_1, v_2 \notin U$ then $(v_1 + U, v_2 + U)$ are linearly independent.

3. Define an explicit isomorphism between $W = \mathbb{R}^2$ and V/U, where $V = \mathbb{R}^5$, $U = \{x \mid x_1 + x_3 + x_5 = 0, x_2 - x_4 = 0\}$. (ie, give linear maps $T : W \to V/U$ and $S : V/U \to W$ that are inverse functions)

4. Prove: let V be finite dimensional, $U \subset V$ a subspace. Then, there exists a subspace $W \subset V$ such that $V = U \oplus W$ and $\pi(W) = V/U$, where $\pi : V \to V/U$ is the quotient map.

5. Consider the quotient space V/U from Exercise 1d). Let $W = \{A \in V \mid A = A^t\}$. Show that the linear map

$$\pi_{|W} : W \to V/U \,;\ w \mapsto w + U$$

is an isomorphism.

6. (Harder) Let $V = P(\mathbb{R})$ and $U = \{p = a_0 + a_1x^2 + a_2x^4 + \dots + a_kx^{2k} \mid k \in \mathbb{N}, a_1, \dots, a_k \in \mathbb{R}\}$. Prove that V/U is infinite dimensional.