Worksheet 9/16. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Linear Maps:

Let $T : V \to W$ be a linear map. Recall that we have the subspaces

$$\operatorname{null} T = \{v \in V \mid T(v) = 0_W\} \subset V,$$

$$\operatorname{range}(T) = \{w \in W \mid w = T(v), \text{ for some } v \in V\} \subset W.$$

1. Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \ ; \ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

   (a) Verify that T is linear.

   (b) Find a basis for $\operatorname{null}(T)$.

   (c) Find a basis for $\operatorname{range}(T)$.

2. Let $T \in L(V, W)$ be a linear map, where V, W are vector spaces. Prove that $\operatorname{range}(T) = \{0\}$ if and only if $\operatorname{null}(T) = V$.

3. Let $T : V \to \mathbb{R}$ be a linear map, where V is a vector space over V. Prove that T is surjective if and only if $T \neq 0 \in L(V, \mathbb{R})$.

4. Suppose that you have a linear map $T : \mathbb{R}^5 \to \mathbb{R}^{10}$. What are the possible values for $\dim \operatorname{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \operatorname{null}(T)$? Define a linear map that realises each of these possible values (Hint: you've already done the work!)

5. Suppose that you have a linear map from $T : \mathbb{R}^7 \to \mathbb{R}^4$. What are the possible values of $\dim \operatorname{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \operatorname{null}(T)$? Define a linear map that realises each of these possible values.

6. Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\operatorname{null}(T) = \operatorname{range}(T)$. Can you do the same for a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$?

7. Let V, W be finite dimensional vector spaces and suppose that $U \subset V$ is a subspace. Does there exist a linear map $T : V \to W$ such that $\operatorname{null}(T) = U$? If you think so, prove it; otherwise, provide a counterexample.

8. Let $T, S \in L(V, W)$ be linear maps. Find a counterexample to the statement: $\operatorname{null}(T + S) = \operatorname{null}(T) + \operatorname{null}(S)$.

9. Let $T, S \in L(V, W)$ be linear maps. Prove that $\operatorname{range}(T + S) \subset \operatorname{range}(T) + \operatorname{range}(S)$. Must these subspaces be equal?

10. (Harder) Let U, V, W be vector spaces and $S \in L(V, W)$. Consider the function

$$f_S : L(U, V) \to L(U, W) \; ; \; T \mapsto S \circ T.$$

Show that f_S is a linear map.

Suppose now that $U = V = \mathbb{R}^2$, $W = \mathbb{R}^3$ and

$$S : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}$$

Show that S and f_S are injective.

What about if we take

$$S : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \underline{x};$$

is S or f_S injective?

Prove, for arbitrary U, V, W and $S \in L(V, W)$: S is injective if and only if f_S is injective.