Worksheet 9/2. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Sums; direct sums

1. Consider the following sums $U + W \subset V$, where $U, W \subset V$ are subspaces (check!). Which of the sums $U+W$ are direct sums? If they are not direct sums find a nonzero vector $v \in U \cap W$ and write an element $z \in U + W$ as $z = u + w = u' + w'$, with $u \neq u'$, $w \neq w'$.

i) $V = \mathbb{R}^3$

$$U = \{\underline{x} \in V \mid x_1 + x_2 + x_3 = 0\},\ W = \{\underline{x} \in V \mid x_1 - x_2 = x_3\}.$$

ii) $V = \mathbb{R}^3$,

$$U = \{(t, 0, -t) \mid t \in \mathbb{R}\},\ W = \{\underline{x} \in V \mid 2x_2 + x_3 = 0\}.$$

iii) $V = \mathbb{C}^3$,

$$U = \{(t, t, t) \mid t \in \mathbb{C}\},\ W = \{(2u, -u, u) \mid u \in \mathbb{C}\}.$$

iv) $V = Mat_2(\mathbb{R})$,

$$U = \{A \in V \mid AB = 0\},\ W = \{A = [a_{ij}] \in V \mid a_{11} + a_{22} = 0\},$$

where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and an arbitrary 2×2 matrix A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

2. Consider the subspace

$$U = \{\underline{x} \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \subset \mathbb{R}^5$$

Find three distinct subspaces $W_1, W_2, W_3 \subset \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W_i$ $(i = 1, 2, 3)$, making sure that you justify how you know that the W_i you have chosen makes the sum $U + W_i$ direct.

Are there a finite number of subspaces $W \subset \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$? Justify your answer.

3. Suppose that $U \subset \mathbb{R}^3$ is a subspace of dimension 2 (recall the notion of dimension from Math 54). Give a criterion to find a subspace $W \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

Suppose now that $U \subset \mathbb{R}^3$ is a subspace of dimension 1. Can you see how to give a criterion to find subspaces $W, R \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W \oplus R$. What should the dimension of W, R be?

4. (*This is for those of you that recall the notion of a basis of $\mathbb{R}^3$. If you can't remember the definition then try this problem again in a week or so...*) Can you use the previous problem to show how to find a basis of $\mathbb{R}^3$, containing $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$?