Worksheet 8/26. Math 110, Fall 2015.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Remark: *In the following we write* $\exp(a) \equiv e^a$.

Complex numbers 1. Row-reduce the following matrices to row-reduced echelon form:

$$A = \begin{bmatrix} 1-i & 2 & -1 \\ -1 & 1+i & 2i \end{bmatrix}, \; B = \begin{bmatrix} -2 & 2i \\ -2i & -2 \end{bmatrix}, \; C = \begin{bmatrix} \exp(i\pi/2) & 1+i \\ -i & \exp(i\pi) \end{bmatrix}$$

2. Let $\omega = \exp(i2\pi/3)$. Show that ω and ω^2 both satisfy the equation $x^3 = 1$. Using this determine all third roots of unity. Can you generalise this result to find all n^{th} roots of unity (ie, all solutions of the equation $x^n = 1$)?

3. Verify that $x^4 - y^4 = (x-y)(x^3 + x^2y + xy^2 + y^3)$. Can you generalise to factor $x^n - y^n$, for arbitrary n, in a similar way?

4. Let $\omega = \exp(i2\pi/n)$. Use the previous exercises to show that

$$1 + x + ... + x^{n-1} = (x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Vector spaces; subspaces

5. Which of the following sets V are vector spaces over F (recall that $F \in \{\mathbb{R}, \mathbb{C}\}$), with the given vector addition and scalar multiplication? You will need to check all of the axioms! (Sorry...)

i) $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \subset \mathbb{R}^3$, with the 'usual' addition of vectors and scalar multiplication inherited from $\mathbb{R}^3$.

ii) $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = \pi\} \subset R^3$, with the 'usual' addition of vectors and scalar multiplication inherited from $\mathbb{R}^3$.

iii) Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and

$$V = \{2 \times 2 \text{ matrices } A \text{ with real entries } \mid AB = 0 \text{ (the zero } 2 \times 2 \text{ matrix)}\} \subset Mat_2(\mathbb{R}),$$

with the 'usual' addition of vectors and scalar multiplication inherited from $Mat_2(\mathbb{R})$, the set of all 2×2 matrices with real entries.

iv) (*Tricky!*) $V = \{(1, y, z) \in \mathbb{R}^3\} \subset \mathbb{R}^3$, where we define vector addition as

$$(1, y, z) + (1, u, v) = (1, y + u, z + v),$$

and scalar multiplication as

$$c \cdot (1, y, z) = (1, cy, cz), \; c \in \mathbb{R}.$$

Note: For this example you will need to say what an appropriate *zero vector* in V should be.

6. All of the above sets are given as subsets of another vector space (which one?). Which of these subsets are subspaces? For those subsets that you think are subspaces prove that they are, in fact, subspaces.

7. Consider the set

$$V = \{e^c \mid c \in \mathbb{R}\} \subset \mathbb{R}^{>0},$$

and define an 'addition'

$$e^c \oplus e^d \stackrel{def}{=} e^{c+d},$$

and a 'scalar multiplication.

$$\lambda \cdot e^c \stackrel{def}{=} e^{\lambda c}.$$

Is V a real vector space when we define addition and scalar multiplication in this way? Prove or explain why not.