Math 110, Fall 2014. The Spectral Theorems

Important things to know:

1. definition of a Hermitian/Euclidean inner product
2. definition of Hermitian/Euclidean space $(V, \langle , \rangle)$
3. definition of an orthonormal basis in a Hermitian/Euclidean space
4. orthonormal bases exist: *this uses Gram-Schmidt process; start with any basis of V and apply Gram-Schmidt*
5. Cauchy-Schwart inequality; Pythagoras' Theorem
6. definition of the adjoint T^+ of an operator $T : V \to V$: *the defining property of T^+ is that $\langle T^+(u), v\rangle = \langle u, T(v)\rangle$, for all $u, v \in V$.*
7. if the matrix of T with respect to an orthonormal basis is A, then the matrix of T^+ with respect to the same orthonormal basis is $\overline{A}^t$.
8. definition of normal/Hermitian/symmetric/anti-Hermitian/anti-symmetric/unitary/orthogonal operators; awareness that all of these concepts depend on the given Hermitian/Euclidean inner product $\langle , \rangle$.
9. orthogonal/unitary/Hermitian/symmetric/anti-Hermitian/anti-symmetric $\Rightarrow$ normal
10. The Complex Spectral Theorem (CST): *Let $(V, \langle , \rangle)$ be a Hermitian space, $T : V \to V$ an operator. Then, T is normal if and only if V admits an orthonormal basis consisting of eigenvectors of T.*
11. Consequences of CST: *if a matrix A is normal (so $A\overline{A}^t = \overline{A}^t A$) then there exists unitary P such that $P^{-1}AP = D$ is diagonal, ie,* **normal if and only if unitarily diagonalisable**
12. Orthogonal Diagonalisation of (anti-)Hermitian Forms: *if $H : \mathbb{C}^n \to \mathbb{C}$; $z \mapsto \sum a_{ij}\overline{z}_i z_j$ is a Hermitian form (resp. anti-Hermitian form) then there exists a unitary change of coordinates $z = Pw$ (with P unitary matrix) such that $H(w) = \lambda_1|w_1|^2 + ... + \lambda_n|w_n|^2$, (resp. $H(w) = i\lambda_1|w_1|^2 + ... + i\lambda_n|w_n|^2$) with $\lambda_1 \geq \cdots \geq \lambda_n$.*
13. The Real Spectral Theorem and its consequences: *any symmetric matrix is orthogonally diagonalisable; if A is an orthogonal matrix then there exists orthogonal P such that $P^{-1}AP$ contains 1×1 blocks on the diagonal and 2×2 blocks of the form $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$. Moreover, these 2×2 blocks correspond to complex (non-real) roots $\alpha \pm i\beta$ of the characteristic polynomial of A.*