Quiz 8. Discussion Section 106. Math 110 Fall 2014.

Name: Solution

1. Determine if the there exists a linear change of coordinates transforming the sesquilinear form $S = i\overline{z}_1 z_1 - 2\overline{z}_1 z_2 + 2\overline{z}_2 z_1$ into $T = 2i\overline{z}_1 z_1 + (1-i)\overline{z}_1 z_2 - (1+i)\overline{z}_2 z_1$.

   Solution: The coefficient matrices of the given forms are

$$A = \begin{bmatrix} i & -2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2i & 1-i \\ -(1+i) & 0 \end{bmatrix}$$

   Since $A = -\overline{A}^t$ and $B = -\overline{B}^t$, the given forms are anti-Hermitian. Let's compute the inertia indices of both; if they are the same then S and T can be transformed into each other by a linear change of coordinates.

   For S we observe that $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ satisfies $S(e_1) = i$; similarly, $T(\frac{1}{\sqrt{2}} e_1) = i$. Hence, $p \geq 1$ for each of S and T. Now, we consider the subspaces

$$\left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid 0 = \overline{e_1}^t A x = i x_1 - 2x_2 \right\} = \text{span}\left(\begin{bmatrix} 1 \\ i/2 \end{bmatrix} \right)$$

$$\left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid 0 = \overline{e_1}^t B x = 2i x_1 + (1-i) x_2 \right\} = \text{span}\left(\begin{bmatrix} -(1+i)/2 \\ 1 \end{bmatrix} \right)$$

   Then, $S([1\ i/2]^t) = -3i$, so that $(p, q) = (1, 1)$ for S, and $T([-(1+i)/2\ 1]^t) = -3i$, so that $(p, q) = (1, 1)$ for T. Hence, there does exist a linear change of coordinates transforming S into T.