1. Consider the quadratic form

\[ Q(x, y) = -x^2 + 8xy + 3y^2. \]

Determine a $2 \times 2$ matrix $A = [a_{ij}]$ and a symmetric bilinear form

\[ B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}; (u, v) \mapsto \sum_{i,j=1}^{2} u_ia_{ij}v_j, \]

such that $Q(x, y) = B((x, y), (x, y))$.

**Solution:** We need $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that

\[ Q(x, y) = B((x, y), (x, y)) = a_{11}x^2 + (a_{12} + a_{21})xy + a_{22}y^2 \implies a_{11} = -1, a_{22} = 3, a_{12} + a_{21} = 8. \]

We must also have $B(u, v) = B(v, u)$ so that $a_{12} = a_{21}$. Hence, We find that $2a_{12} = 8 \implies a_{12} = a_{21} = 4$. Thus, the matrix is

\[ A = \begin{bmatrix} -1 & 4 \\ 4 & 3 \end{bmatrix}, \]

and

\[ B(u, v) = -u_1v_1 + 4u_1v_2 + 4u_2v_1 + 3u_2v_2. \]

You can also use the formula

\[ B(u, v) = \frac{1}{2} (Q(u_1 + v_1, u_2 + v_2) - Q(u_1, u_2) - Q(v_1, v_2)). \]