Quiz 2. Discussion Section 103. Math 110 Fall 2014.

Name: Solution

1. Consider the quadratic form $Q = 4x^2 + \sqrt{3}xy + 3y^2$. Determine a change of coordinates

$$\begin{aligned} x &= \alpha u + \beta v \\ y &= \gamma u + \delta v \end{aligned}$$

and find $\lambda, \mu \in \mathbb{R}$, such that $Q = \lambda u^2 + \mu v^2$.

Solution: (1) We use the formula from the book to rotate the coordinate system by θ, where $\cot 2\theta = (4-3)/\sqrt{3} = 1/\sqrt{3}$. Hence, we must have $2\theta = \pi/3 \implies \theta = \pi/6$. Then, the deseired change of coordinates is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \implies \begin{aligned} x &= \tfrac{\sqrt{3}}{2}u - \tfrac{1}{2}v \\ y &= \tfrac{1}{2}u + \tfrac{\sqrt{3}}{2}v \end{aligned}$$

Thus, we have

$$\begin{aligned} 4x^2 + \sqrt{3}xy + 3y^2 =& (3u^2 - 2\sqrt{3}uv + v^2) + \frac{\sqrt{3}}{4}(\sqrt{3}u^2 + 2uv - \sqrt{3}v^2) \\ &+ \frac{3}{4}(u^2 + 2\sqrt{3}uv + 3v^2) \\ =& \frac{9}{2}u^2 + \frac{5}{2}v^2 \end{aligned}$$

(2) Alternatively, you can proceed as follows: completing the square gives

$$4x^2 + \sqrt{3}xy + 3y^2 = 4(x - \frac{\sqrt{3}}{8}y) + \frac{45}{16}y^2.$$

Let $u = x - \frac{\sqrt{3}}{8}y, v = y$. Then, $x = u + \frac{\sqrt{3}}{8}y, v = y$, and

$$-x^2 + \sqrt{3}xy - 2y^2 = 4u^2 + \frac{45}{16}v^2.$$

Note: there are many solutions to this problem - I did not specify what type of change of coordinates had to be used.