Quiz 11. Discussion Section 106. Math 110 Fall 2014.

Name: Solution

1. Consider the following permutation matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $X = -I_3 - 2A + 2A^2$, where I_3 is the 3×3 identity matrix. Determine the eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and the corresponding eigenspaces.

Solution: The given matrix satisfies $A^3 = I_3$ (notice that A is a permutation matrix). If λ is an eigenvalue of A and $Av = \lambda v$, with $v \neq 0$, then

$$Xv = (-I_3 - 2A + 2A^2)v = -v - 2Av + 2A^2v = (-1 - 2\lambda + 2\lambda^2)v.$$

Here we have used the fact: if $Av = \lambda v$, with $v \neq 0$, then $A^n v = \lambda^n v$.

Hence, if λ is an eigenvalue of A then $(-1 - 2\lambda + 2\lambda^2)$ is an eigenvalue of X. You can check that A has characteristic polynomial $-\lambda^3 + 1$, so it's eigenvalues are $\lambda_1 = 1, \lambda_2 = e^{2\pi i/3}, \lambda_3 = e^{4\pi i/3}$ (the cube roots of unity). Also, you can show that (by row-reduction)

$$\text{nul}\,(A - \lambda_1 I_3) = \text{span}((1,1,1)^t),\ \text{nul}\,(A - \lambda_2 I_3) = \text{span}((\lambda_1, \lambda_1^2, 1)^t),$$

$$\text{nul}\,(A - \lambda_2 I_3) = \text{span}((\lambda_2, \lambda_2^2, 1)^t).$$

Hence, X must have the eigenvalues $-1 - 2\lambda_i + 2\lambda_i^2$, for $i = 1, 2, 3$, and with the same eigenspaces as above.