Quiz 10. Discussion Section 106. Math 110 Fall 2014.

Name: Solution

1. Consider the following permutation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The matrix A defines orthogonal/unitary (*you don't have to show this*) transformations

$$T : \mathbb{R}^4 \to \mathbb{R}^4,\ x \mapsto Ax, \quad S : \mathbb{C}^4 \to \mathbb{C}^4, x \mapsto Ax.$$

What are the real/complex normal forms of T, S? That is, determine matrices B, C, such that

$$U^t A U = B, \quad \overline{V}^t A V = C,$$

where U, V are orthogonal/unitary matrices. (Hint: use the real spectral theorem)

Solution: Since S is a unitary operator (its matrix with respect to the (orthonormal) standard basis is A and $A^{-1} = \overline{A}^t$) it is diagonalisable - hence, we need to determine the eigenvalues of A. The characteristic polynomial of A is $(\lambda^3 - 1)(\lambda - 1) = (\lambda - 1)^2(\lambda - \mu)(\lambda - \mu^2)$, where $\mu = e^{2\pi i/3}$ is the cube root of unity with positive imaginary part. Hence, we take (put a 0 everywhere left blank)

$$C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \mu & \\ & & & \mu^2 \end{bmatrix}$$

T is an orthogonal operator (because its matrix with respect to the (orthonormal) standard basis is A and $A^{-1} = A^t$) we can use the real spectral theorem to determine the normal form for T. This depends on finding the eigenvalues of A. As discovered above, they are $1, 1, \mu, \mu^2$ - notice that $\mu^2 = \overline{\mu}$ (because $\mu^2 = \mu^{-1} = \overline{\mu}/|\mu|^2 = \overline{\mu}$). Hence, the normal form we take for B is

$$B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \alpha & -\beta \\ & & \beta & \alpha \end{bmatrix}$$

where $\alpha = Re(\mu)$, $\beta = Im(\mu)$. Since $\mu = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2}(1 + \sqrt{3}i)$, we have $\alpha = -1/2, \beta = \sqrt{3}/2$.