Math 110, Fall 2014. Quadratic/Hermitian Form Practice Problems

1. Consider the following Hermitian forms $H : \mathbb{C}^2 \to \mathbb{R}$ (where $\mathbb{C}^2$ is the standard Hermitian space, with Hermitian inner product $\overline{z}_1 z_1 + \overline{z}_2 z_2$)

$$H(z) = 2i\overline{z}_1 z_2 - 2i\overline{z}_2 z_1, \quad H(z) = -|z_1|^2 - i\overline{z}_1 z_2 + i\overline{z}_2 z_1 + |z_1|^2,$$
$$H(z) = |z_1|^2 + (1-i)\overline{z}_1 z_2 + (1+i)\overline{z}_2 z_1 + |z_2|^2.$$

Answer the following questions:

(a) Determine the normal form of H up to a linear change of coordinates.

(b) Determine the normal form of H up to a unitary change of coordinates.

(c) Determine a unitary change of coordinates $z = Pw$ such that $H(w) = \lambda_1 |w_1|^2 + \lambda_2 |w_2|^2$, with $\lambda_1 \geq \lambda_2$.

(d) For which of the above Hermitian forms does there exist a linear change of coordinates transforming one into the other?

Solution: The matrices of the above Hermitian forms are

$$\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1-i \\ 1+i & 1 \end{bmatrix}$$

The characteristic polynomials of these matrices are

$$x^2 - 4, \quad x^2 - 2, \quad x^2 - 2x - 1$$

Since each of these polynomials has one positive and one negative root, the normal form (up to any linear change of coordinates) of the Hermitian forms are

$$|w_1|^2 - |w_2|^2.$$

The normal form (up to a unitary change of coordinates) are

$$2|w_1|^2 - 2|w_2|^2, \quad \sqrt{2}|w_1|^2 - \sqrt{2}|w_2|^2, \quad (1+\sqrt{2})|w_1|^2 - (\sqrt{2}-1)|w_2|^2.$$

To find a unitary change of coordinates we need to determine an orthonormal basis of eigenvectors for each of the above matrices: we find the following bases of eigenvectors (you need to scale them by their length with respect to standard Hermitian form)

$$\left(\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right), \quad \left(\begin{bmatrix} 1-\sqrt{2} \\ -i \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} \\ -i \end{bmatrix},\right), \quad \left(\begin{bmatrix} \sqrt{2} \\ 1+i \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ 1+i \end{bmatrix}\right)$$

Each of the Hermitian forms can be transformed into the other by a linear change of coordinates because they all have the same normal form (up to a linear change of coordinates).

2. Consider the following quadratic forms $Q : \mathbb{R}^3 \to \mathbb{R}$, where $\mathbb{R}^3$ is the standard Euclidean space,

$$Q(x) = -x_1^2 + 2x_2^2 + 2x_1x_3 - x_3^2, \quad Q(x) = -2x_1x_2 + 2x_1x_3 + x_2^2,$$
$$Q(x) = x_1^2 + x_1x_2 + x_2x_3 + x_1x_3 + x_3^2$$

(a) Determine the normal form of Q up to a linear change of coordinates.

(b) (THIS PROBLEM HAS BEEN CHANGED! I MADE A MISTAKE) Determine the normal form of the first quadratic form Q up to a orthogonal change of coordinates.

(c) Determine an orthogonal change of coordinates $x = Pu$ such that the first quadratic form Q takes the form $Q(u) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2$.

(d) Which of the following hypersurfaces in $\mathbb{R}^3$ can be transformed into each other by a linear change of coordinates?

$$A = \{x \mid -x_1^2 + 2x_2^2 + 2x_1x_3 - x_3^2 = 1\}, \quad B = \{x \mid -2x_1x_2 + 2x_1x_3 + x_2^2 = -1\},$$

$$C = \{x \mid x_1^2 + x_1x_2 + x_2x_3 + x_1x_3 + x_3^2 = 1\}$$

Solution: The matrices of the above quadratic forms are

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

We can't use Sylvester's rule for the first form since its matrix is not invertible. For the second form, if we make the swap $x_1 \leftrightarrow x_2$ then the quadratic form becomes $x_1^2 - 2x_1x_2 + 2x_2x_3$, with matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying Sylvester's rule to this matrix: let Δ_i be the determinant of the upper-left $i \times i$ matrix. Then, we obtain the sequence $(1, 1, -1, -1)$. There is one sign change in this sequence so that $q = 1, p = 2$. Hence, the normal form is

$$u_1^2 + u_2^2 - u_3^2.$$

Using Sylvester's rule for the last matrix gives the sequence $(1, 1, -1/4, -1/4)$, so that $q = 1, p = 2$ and the normal form is

$$u_1^2 + u_2^2 - u_3^2.$$

The rank of the fist matrix is $2 = p + q$. Since Q is negative on $\text{span}(e_1) - \{0\}$ then $q \geq 1$; since Q is positive on $\text{span}(e_2)$ then $p \geq 1$. Hence, we must have $p = q = 1$. So, the normal form of the first form (up to a linear change of coordinates) is

$$u_1^2 - u_2^2.$$

To determine the normal forms up to an orthogonal change of coordinates we need to determine the characteristic polynomials: we find $-x(x^2 - 4)$. Hence, the normal form (up to an orthogonal change of coordinates) is

$$2u_1^2 - 2u_2^2.$$

We need to find an orthonormal basis of eigenvectors of A in order to determine the required P: we find that we can take

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Since only the second and third quadratic forms have the same normal forms up to linear change of coordinates, these are the only potential hypersurfaces that can be transformed into each other. However, if Q_2 is the second quadratic form, then $B = \{x \mid -Q_2(x) = 1\}$. Since $-Q_2$ has inertia indices $(p, q) = (1, 2)$ (note the change!) then none of the given hypersurfaces can be transformed into each other.