Math 110, Fall 2014. Quadratic/Hermitian Form Practice Problems

1. Consider the following Hermitian forms $H : \mathbb{C}^2 \to \mathbb{R}$ (where $\mathbb{C}^2$ is the standard Hermitian space, with Hermitian inner product $\overline{z}_1 z_1 + \overline{z}_2 z_2$)

$$H(z) = 2i\overline{z}_1 z_2 - 2i\overline{z}_2 z_1, \quad H(z) = -|z_1|^2 - i\overline{z}_1 z_2 + i\overline{z}_2 z_1 + |z_1|^2,$$

$$H(z) = |z_1|^2 + (1-i)\overline{z}_1 z_2 + (1+i)\overline{z}_2 z_1 + |z_2|^2.$$

Answer the following questions:

(a) Determine the normal form of H up to a linear change of coordinates.

(b) Determine the normal form of H up to a unitary change of coordinates.

(c) Determine a unitary change of coordinates $z = Pw$ such that $H(w) = \lambda_1 |w_1|^2 + \lambda_2 |w_2|^2$, with $\lambda_1 \geq \lambda_2$.

(d) For which of the above Hermitian forms does there exist a linear change of coordinates transforming one into the other?

2. Consider the following quadratic forms $Q : \mathbb{R}^3 \to \mathbb{R}$, where $\mathbb{R}^3$ is the standard Euclidean space,

$$Q(x) = -x_1^2 + 2x_2^2 + 2x_1x_3 - x_3^2, \quad Q(x) = -2x_1x_2 + 2x_1x_3 + x_2^2,$$

$$Q(x) = x_1^2 + x_1x_2 + x_2x_3 + x_1x_3 + x_3^2$$

(a) Determine the normal form of Q up to a linear change of coordinates.

(b) Determine the normal form of the first two quadratic forms Q up to a orthogonal change of coordinates.

(c) Determine an orthogonal change of coordinates $x = Pu$ such that the first quadratic form Q takes the form $Q(u) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2$.

(d) Which of the following hypersurfaces in $\mathbb{R}^3$ can be transformed into each other by a linear change of coordinates?

$$A = \{x \mid -x_1^2 + 2x_2^2 + 2x_1x_3 - x_3^2 = 1\}, \quad B = \{x \mid -2x_1x_2 + 2x_1x_3 + x_2^2 = -1\},$$

$$C = \{x \mid x_1^2 + x_1x_2 + x_2x_3 + x_1x_3 + x_3^2 = 1\}$$