Math 110, Fall 2014. Quadratic Forms

Consider the quadratic form

$$Q : \mathbb{R}^n \to \mathbb{R} \; ; \; \underline{x} \mapsto \sum_{1 \le i \le j \le n} a_{ij} x_i x_j.$$

It is a Theorem that we can find a change of coordinates $\underline{x} = C\underline{u}$ such that, in this new coordinate system,

$$Q(\underline{u}) = \lambda_1 u_1^2 + ... + \lambda_{p+q} u_{p+q}^2,$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0$, and $0 > \lambda_{p+1} \ge \cdots \ge \lambda_{p+q}$.

We can go one better than what we have above: we can find a change of coordinates $x = Dz$ such that, in this new coordinate system,

$$Q(\underline{z}) = z_1^2 + ... + z_p^2 - z_{p+1}^2 - ... - z_{p+q}^2.$$

We call (p, q) the **inertia indices**. They are defined in a more intrinsic way: we have

$$p = \max\{\dim U \mid U \subset \mathbb{R}^n \text{ is a subspace and } Q > 0 \text{ on } U - \{0\}\},$$

$$q = \max\{\dim U \mid U \subset \mathbb{R}^n \text{ is a subspace and } -Q > 0 \text{ on } U - \{0\}\},$$

For example,

1. Consider the quadratic form $Q(\underline{x}) = x_1^2 - 2x_1x_2 + x_3^2$ on $\mathbb{R}^3$. Let's determine the inertia indices: completing the square, we see that

   $$Q = (x_1 - x_2)^2 - x_2^2 + x_3^2.$$

   Thus, we see by inspection that Q is positive on the subspace $\left\{ \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$, so that $p \ge 2$. We must also have $p \le 3$ (since Q is a form on $\mathbb{R}^3$). As $Q(e_1 + e_2) = (1-1)^2 - 1^2 + 0^2 = -1 < 0$, we do not have that $Q > 0$ everywhere away from the origin, so that $p < 3$ and we must have $p = 2$ (here e_1, e_2 are standard basis vectors in $\mathbb{R}^3$).

   Similarly, we see that $-Q$ is positive away from the origin on the subspace $\left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$.

   Hence, $q \ge 1$. As $p + q \le 3$, we must have $q = 1$.

2. Consider the quadratic form $Q(\underline{x}) = 2x_1x_2$ on $\mathbb{R}^3$. Completing the square we find

   $$Q = \frac{1}{2}\left((x_1 + x_2)^2 - (x_1 - x_2)^2\right)$$

   By inspection, we see that Q is positive on the subspace $\left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$. Thus, $p \ge 1$.

   Since $Q(e_1) = 0$, Q is not positive everywhere away from the origin so that $p < 3$ (as argued above). Suppose that Q is positive on a subspace U with $\dim U = 2$, and let $u, v \in U$ be a basis (we are going to show this is impossible). We have

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Thus, for every λ, μ (so that $(\lambda, \mu) \neq (0, 0)$), we have $Q(\lambda u + \mu v) > 0$. That is,

$$Q(\lambda u + \mu v) = 2(\lambda u_1 + \mu v_1)(\lambda u_2 + \mu v_2) > 0.$$

In particular, note that Q would be positive on the subspace $\text{span}(e_1, e_2) \subset \mathbb{R}^3$. However, $Q(e_1) = 0$ so this does not hold. Hence, it can't be the case that Q is positive on a subspace U with $\dim U = 2$, and therefore, $p = 1$.

Note that $-Q$ is positive on the subspace $\left\{ \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$, so that $q \geq 1$. As $p+q \leq 3$ we must have $q \leq 2$. In a similar way to our determination of p, we must have $p \neq 2$ (if U was a two dimensional subspace on which $-Q$ is positive, then $-Q$ would be positive on $\text{span}(e_1, e_2)$). Hence, $q = 1$.

We can determine that $p, q \neq 2$ using the concept of the **rank** of a quadratic form: given the quadratic form Q, we first determine the coefficient matrix of Q; recall that this is the matrix B such that the symmetric bilinear form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (using Givental's abuse of notation) takes the form $Q(x, y) = x^t B y$, and the quadratic form (careful!) takes the form $Q(x) = x^t B x$. In general, the ij term of the coefficient matrix B of the quadratic form $Q = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$ is

$$b_{ij} = \begin{cases} a_{ij}, & \text{when } i = j, \\ \frac{1}{2} a_{ij}, & \text{when } i \neq j. \end{cases}.$$

The matrices of the examples above are

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of the quadratic form is then the rank of the matrix B. The rank of a quadratic form can be intepreted as

$$\text{rank} Q = \max\{U \subset \mathbb{R}^n \mid U \text{ is a subspace and } Q = 0 \text{ on } U\}.$$

The main result is that $p + q = \text{rank} B$; this is in Givental's notes, although perhaps not explicitly stated. Hence, for the second example above we can see that $\text{rank} B = 2$ so that $p + q = 2$, and since it is not too hard to show that $p, q \geq 1$, we must have $p = q = 1$.

Finding 'orthogonal' bases of Q:

!! BEWARE !!: these bases are not 'orthogonal' in the usual sense (ie, with respect to the dot product on $\mathbb{R}^n$)

When you complete the square for Q, and when $p + q = n$, you are, in fact, determining bases $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ such that $Q(v_i, v_j) = 0$, for $i \neq j$. **However, sometimes care is needed!** This is best illustrated by some examples:

1. Consider the quadratic form $Q = x_1^2 - 2x_1x_3 + 4x_2x_3 - x_2^2$. Completing the square gives

$$Q = (x_1 - x_3)^2 - x_2^2 + 4x_2x_3 - x_3^2 = (x_1 - x_3)^2 - (x_2 - 2x_3)^2 + 3x_3^2$$

If we set

$$u_1 = x_1 - x_3,\ u_2 = x_2 - 2x_3,\ u_3 = x_3,$$

then $Q(u) = u_1^2 - u_2^2 + 3u_3^2$. Note that the above equations can be written in matrix form as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Denoting the matrix A^{-1}, we find its inverse is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, we see that, for every $u \in \mathbb{R}^3$

$$u^t A^t B A u = u^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} u.$$

Here, $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ is the coefficient matrix of Q (with respect to the standard coordinates). The above formula thus implies that A^tBA is a diagonal matrix, so that the columns of A are a Q-orthogonal basis (this is discussed in Givental's notes).

Note: the columns of A are NOT orthogonal with respect to the usual notion of orthogonality (via the dot product on $\mathbb{R}^3$); but this is OK!

Since $p + q = 3$, we could have obtained a matrix A such that A^tBA is diagonal as follows: let U_+ be a subspace such that $\dim U_+ = p$ and $Q > 0$ on U_+; similarly, let U_- be a subspace such that $\dim U_- = q$ and $-Q > 0$ on U_-. We see that we can take

$$U_+ = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2 - 2x_3 = 0 \right\} = \left\{ \begin{bmatrix} x \\ 2y \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

$$U_- = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 - x_3 = 0,\ x_3 = 0 \right\} = \left\{ \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

Now, we find a basis of U_+ and U_- as follows: since $\dim U_- = 1$ we just take any nonzero vector in U_-; for example e_2. For U_+, choose $v_1 \in U_+$ such that $Q(v_1) > 0$, say $v_1 = e_1$. Then, consider those elements $v \in U_+$ satisfying

$$0 = Q(v, v_1) = x_1 - x_3.$$

Thus, we want an element in U_+ satisfying both $x_2 - 2x_3 = 0$ and $x_1 - x_3 = 0$; this is precisely the subspace $\left\{ \begin{bmatrix} x \\ 2x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}$. Thus, we can recover the columns of A above in this way.

2. Consider the quadratic form $Q = 4x_1x_2 + 4x_1x_3 = 4x_1(x_2 + x_3)$. We complete the square to get

$$Q = (x_1 + (x_2 + x_3))^2 - (x_1 - (x_2 + x_3))^2.$$

Hence, we can set

$$u_1 = x_1 + x_2 + x_3, \; u_2 = x_1 - x_2 - x_3,$$

but what about u_3? It's not so clear how to proceed...

We need to proceed as we did in the previous example: let's find subspaces U_+, U_- as above. First observe that the rank of Q is 2 (by looking at the coefficient matrix of Q). Thus, we firstly need to take a basis of $\ker Q$. It can be seen that this is

$$\ker Q = \text{nul}\left(\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 0 \\ x \\ -x \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

The remaining basis vectors must be Q-orthogonal to $\ker Q$; so we must ensure that vectors in U_+ and U_- are Q-orthogonal to $\ker Q$. Any such vector x must satisfy $2x_2 - 2x_3 = 0$, because we require $Q(x, e_2 - e_3) = 0$. Then, we can take

$$U_+ = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 = x_2 + x_3, x_2 - x_3 = 0 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2x \\ x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

$$U_- = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 = -(x_2 + x_3), x_2 - x_3 = 0 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -2x \\ x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

So, if we take

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

then

$$A^t BA = \begin{bmatrix} 16 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Consider the form $Q = x_1x_2 + x_1x_3 + x_2x_3$. For this example, we need a different approach: we follow the (implicit) algorithm discussed in Givental's notes. First, we observe that the coefficient matrix of Q is

$$B = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$