Math 110, Fall 2014. ODE Practice Problems

Determine the general solution to the following systems of linear ODE:

1.
$$\begin{aligned} x_1'(t) &= 2x_1(t) - x_3(t) \\ x_2'(t) &= x_2(t) + x_3(t) \\ x_3'(t) &= x_2(t) \end{aligned}$$

2.
$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(t) + x_3(t) \\ x_2'(t) &= x_1(t) + x_2(t) + x_3(t) \\ x_3'(t) &= -2x_1(t) - 2x_2(t) - 2x_3(t) \end{aligned}$$

3.
$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(t) \\ x_2'(t) &= x_2(t) + x_3(t) \\ x_3'(t) &= -x_2(t) - x_3(t) \end{aligned}$$

We need to find the matrix form of the above systems of ODEs: we have

$$x'(t) = Ax(t),$$

where A is one of the following matrices

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

We need to determine the Jordan forms of these matrices: we must first obtain the characteristic polynomials

$$(2-x)(x^2-x-1), \quad x^3, \quad (1-x)x^2$$

The first polynomial has three distinct roots $(1, (1 \pm \sqrt{5})/2)$ so its Jordan form is diagonal with these roots appearing on the diagonal.

The second matrix is nilpotent and, since $\dim \operatorname{nul}(A) = 2$, the Jordan form has two 0-Jordan blocks. Hence, the Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For the third matrix, we note that $\dim \operatorname{nul}(A) = 1$ so that the Jordan form contains only one 0-Jordan block. Hence, the Jordan form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $P = [u_1 \ u_2 \ u_3]$ is a matrix such that $P^{-1}AP = J$ is the Jordan form, then the general solution to the given system of ODEs is

$$x(t) = e^{At}x(0) = Pe^{Jt}P^{-1}x(0)$$

For the first system we have

$$e^{Jt} = \begin{bmatrix} e^{2t} & & \\ & e^{(1+\sqrt{5})t/2} & \\ & & e^{(1-\sqrt{5})t/2} \end{bmatrix}$$

and, the general solution is

$$x(t) = c_1 e^{2t} u_1 + c_2 e^{(1+\sqrt{5})t/2} u_2 + c_3 e^{(1-\sqrt{5})t/2} u_3.$$

For the second system we find that (because $J^2 = 0$)

$$e^{Jt} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, we find

$$x(t) = c_1(1 + t)u_1 + c_2 u_2 + c_3 u_3.$$

To find suitable u_1, u_2, u_3 we proceed as follows: u_2 must be a vector that does not lie in $\mathrm{nul}(A) = \mathrm{span}(e_1 - e_2, e_1 - e_3)$. Hence, we can take $u_2 = e_1$, so that $u_1 = e_1 + e_2 - 2e_3$. Then, we need $u_3 \in \mathrm{nul}(A)$ such that (u_1, u_3) is linearly independent: we can take $u_3 = e_1 - e_2$.

For the third system we find

$$e^{Jt} = \begin{bmatrix} e^{t} & & \\ & 1 & t \\ & & 1 \end{bmatrix}$$

and the general solution is

$$x(t) = c_1 e^{t} u_1 + c_2(1 + t)u_2 + c_3 u_3,$$

where u_1 is an eigenvector with eigenvalue 1. We can take $u_1 = e_1$. u_3 must be a vector such that $u_3 \notin \mathrm{nul}(A) + \mathrm{span}(e_1) = \mathrm{span}(e_1 - e_2, e_1) = \mathrm{span}(e_1, e_2)$. Hence, we can take $u_3 = e_3$. Then, $u_2 = Au_3 = e_1 - e_2$.