Math 110, Fall 2014. Some Jordan Form Examples

Here are the facts that we are going to repeatedly use :

- $\dim \operatorname{null}(T - \lambda)$ equals the number of λ-Jordan blocks appearing.
- if T admits precisely one eigenvalue λ, and e is the smallest integer such that $\operatorname{null}(T - \lambda)^e = V$, then the size of the largest λ-Jordan block is $e \times e$.

Problem: Determine the Jordan form and Jordan bases for the following operators:

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \ ; \ \underline{x} \mapsto A\underline{x},$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

i) Since A is upper-triangular with 1s on the diagonal then the only eigenvalue is $\lambda = 1$. Since $\dim \operatorname{null}(T - I) = 2$ we have that the Jordan form of T has two 1-Jordan blocks. Since $(T - I)^3 = 0 \in L(\mathbb{C}^4)$, and $(T - I)^2 \neq 0$, then we see that the largest 1-Jordan block is 3×3. The only possible Jordan form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A Jordan basis is a basis (v_1, v_2, v_3, v_4) such that

$$T(v_1) = v_1, \ T(v_2) = v_2 + v_1, \ T(v_3) = v_3 + v_2, \ T(v_4) = v_4.$$

In particular, we see that $(T - I)^3(v_3) = 0$, while $(T - I)^2(v_3) \neq 0$. Moreover, once we determine v_3 we find that $v_2 = (T - I)(v_3)$, and $v_1 = (T - I)(v_2)$. We compute

$$\operatorname{null}((T - I)^2) = \operatorname{span}(e_1, e_2, e_3),$$

so that we can choose $v_3 = e_4$. Hence, $v_2 = (T - I)(v_3) = e_1 + e_2$, $v_1 = (T - I)(v_2) = e_1$. Now, we need $v_4 \in \operatorname{null}((T - I)) = \operatorname{span}(e_1, -e_2 + e_3)$ such that (v_1, v_2, v_3, v_4) is linearly independent. So, we can take $v_4 = -e_2 + e_3$. Then,

$$(e_1, e_1 + e_2, e_4, -e_2 + e_3),$$

is a Jordan basis.

ii) Since A is upper-triangular with 0s on the diagonal then the only eigenvalue is $\lambda = 0$. Since $\dim \operatorname{null} T = 2$ we have that the Jordan form of T has two Jordan blocks. Since $T^2 = 0 \in L(\mathbb{C}^4)$, while $T \neq 0$, then the largest Jordan block is 2×2. The only possible Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A Jordan basis is a basis (v_1, v_2, v_3, v_4) such that

$$T(v_1) = 0,\ T(v_2) = v_1,\ T(v_3) = 0,\ T(v_4) = v_3.$$

In particular, we have $T^2(v_2) = 0$ while $T(v_2) = v_1 \neq 0$, and $T^2(v_4) = 0$, while $T(v_4) = v_3 \neq 0$. Since $\operatorname{null} T = \operatorname{span}(e_1, e_2)$ and $\operatorname{null} T^2 = \mathbb{C}^4$, then we can take $v_2 = e_3$ and $v_4 = e_4$. Then, $v_1 = T(v_2) = e_2$ and $v_3 = T(v_4) = e_1 + e_2$. Hence, a Jordan basis is

$$(e_2, e_3, e_1 + e_2, e_4).$$

iii) Since A is lower-triangular with 02s on the diagonal then the only eigenvalue is $\lambda = 2$. Since $\dim \operatorname{null}(T - 2) = 2$ we have that the Jordan form of T has two Jordan blocks. Since $(T-2)^3 = 0 \in L(\mathbb{C}^4)$ and $(T-2)^2 \neq 0$, the largest Jordan block is 3×3 and the only possible Jordan form is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Proceeding as in i) above, and using that $\operatorname{null}(T - 2)^2 = \operatorname{span}(e_2, e_3, e_4)$, we can take $v_3 = e_1$, so that $v_2 = (T - 2)(v_3) = -e_2 - e_3 + e_4$ and $v_1 = (T - 2)(v_2) = -e_4$. As $\operatorname{null}((T - 2)) = \operatorname{span}(e_3, e_4)$, if we take $v_4 = e_3$ then

$$(-e_4, -e_2 - e_3 + e_4, e_1, e_3),$$

is a Jordan basis.

iv) You can check that $T^2 = 0 \in L(\mathbb{C}^4)$ (since $A^2 = 0$) so that $\operatorname{null}(T^2) = \mathbb{C}^4$. Hence, the largest Jordan block is 2×2 and the only possible Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proceeding as in ii) above, and using that $\operatorname{null} T = \operatorname{span}(e_1 + e_4, e_2 - e_3)$ we can take $v_2 = e_1$ and $v_4 = e_2$ so that $v_1 = T(v_2) = e_1 + e_4$ and $v_3 = T(v_4) = -e_2 + e_3$. Then,

$$(e_1 + e_4, e_1, -e_2 + e_3, e_2),$$

is a Jordan basis.

Note that we have to be careful here - we need to ensure that we choose v_2, v_4 which are linearly independent and such that $\operatorname{span}(v_2, v_4) \cap \operatorname{null} T = \{0\}$ (ie, if we chose $v_4 = e_4$ then $v_2 + v_4 = e_1 + e_4 \in \operatorname{null} T$).