Math 110, Fall 2013. Midterm Review

Things You Must Know:

Definitions:

Ch. 1 - vector space over F, subspace, sums of subspaces, direct sums of subspace (in particular, they are special types of sums).

Ch. 2 - finite dimensional spaces, spanning lists, linearly (in)dependent lists, basis, dimension.

Ch. 3 - linear maps, null(T), range(T), $L(V, W)$, $L(V)$, operators, matrix of a linear map, injective, surjective, bijective, invertible linear map, isomorphic.

Ch. 4 - Fundamental Theorem of Algebra, degree of polynomials, division algorithm.

Ch. 5 - invariant subspaces, upper-triangular matrices, eigenvalues, eigenvectors, diagonal matrices.

Theorems:

Ch. 1 -

- $U \subset V$ is a subspace if and only if $\forall \lambda, \mu \in F$, $u, v \in U$ we have $\lambda u + \mu v \in U$.
- for $U, W \subset V$ subspaces, $U + W$ is a subspace.
- $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.

Ch. 2 -

- if V is f.d and $(v_1, \dots, v_m)$ is linearly dependent in V, and $v_1 \neq 0$ then there is $j \in \{2, \dots, m\}$ such that $(v_1, \dots, v_i)$ is linearly independent, for $i = 1, \dots, j-1$, and $v_j \in \text{span}(v_1, \dots, v_{j-1})$. Hence, $\text{span}(v_1, \dots, v_j) = \text{span}(v_1, \dots, v_{j-1})$.
- if V is f.d. with $V = \text{span}(v_1, \dots, v_m)$, and $(v_1, \dots, v_k)$ is linearly independent, then $k \leq m$.
- if $\dim V = n$ and $(v_1, \dots, v_k)$ is linearly independent then $k \leq n$. (*linearly independent lists are 'small'*)
- if $\dim V = n$ and $(v_1, \dots, v_m)$ is a spanning list then $m \geq n$. (*spanning lists are 'big'*)
- bases exist!
- every linearly independent list of length $\dim V$ is a basis.
- every spanning list of length $\dim V$ is a basis.
- every linearly independent list can be extended to a basis.
- if $(v_1, \dots, v_m)$ is a spanning list in V, $\dim V = n$, then there are $i_1, \dots, i_n \in \{1, \dots, m\}$ distinct, such that $(v_{i_1}, \dots, v_{i_n})$ is a basis.
- if $U \subset V$ is a subspace then $\dim U \leq \dim V$. In particular, any basis of U can be extended to a basis of V.
- (**Dimension Formula**) if $U, W \subset V$ are subspaces, V f.d., then

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W.$$

- $U + W$ is a direct sum if and only if $\dim(U + W) = \dim U + \dim W$.

Ch. 3 -

- let $T \in L(V, W)$. Then, $\text{null}(T) \subset V$ is a subspace; $\text{range}(T) \subset W$ is a subspace.

- $T \in L(V, W)$ is injective if and only if $\text{null}(T) = \{0\}$.
- $T \in L(V, W)$ is surjective if and only if $\text{range}(T) = W$.
- let V be f.d., $T \in L(V, W)$. Then, $\dim V = \dim \text{null}(T) + \dim \text{range}(T)$.
- if $\dim V > \dim W$ then no linear map $T \in L(V, W)$ is injective.
- if $\dim V < \dim W$ then no linear map $T \in L(V, W)$ is surjective.
- if $T \in L(V, W)$ is injective and $(v_1, \dots, v_k) \subset V$ is linearly independent then $(T(v_1), \dots, T(v_k)) \subset W$ is linearly independent.
- if $T \in L(V, W)$ is surjective and $(v_1, \dots, v_m)$ spans V then $(T(v_1), \dots, T(v_m))$ spans W.
- $L(V, W)$ is a vector space over F; it had $\dim L(V, W) = \dim V \dim W$
- $L(V)$ is a vector space over F.
- $T \in L(V)$ is injective if and only T is surjective if and only if T is invertible.
- V and W are isomorphic if and only if $\dim V = \dim W$.
- if $(v_1, \dots, v_n) \subset V$ is a basis of V and $w_1, \dots w_n \in W$ arbitrary, then there is a unique linear map $T \in L(V, W)$ such that $T(v_i) = w_i$.

Ch. 4 -

- every nonconstant polynomial with complex coefficients admits a root.
- every nonconstant polynomial with complex coefficients of degree m has exactly m roots.
- if q divides p and $\deg q > \deg p$ the $p = 0$ is the zero polynomial.
- every polynomial with real coefficients and odd degree has a real root.

Ch. 5 -

- if $v_1, \dots, v_m$ are eigenvectors of $T \in L(V)$, corresponding to distinct eigenvalues, thatn $(v_1, \dots, v_m)$ is linearly independent.
- any $T \in L(V)$ has at most $\dim V$ distinct eigenvalues.
- every $T \in L(V)$, for V a complex vector space, admits an eigenvalue.
- for any $T \in L(V)$, V a complex vector space, there is a basis of V, $B = (v_1, \dots, v_n)$ such that $[T]_B$, the matrix of T relative to B, is upper-triangular. In particular, $T(v_i) \in \text{span}(v_1, \dots, v_i)$, for each i, and $\text{span}(v_1, \dots, v_i)$ is T-invariant, for each i.
- Suppose that the matrix of T relative to the basis B is upper-triangular. Then, the eigenvalues of T are precisely the diagonal entries of this matrix.
- $T \in L(V)$ has a diagonal matrix with respect to some basis of V if and only if V admits a basis of eigenvectors.
- $T \in L(V)$ has a diagonal matrix with respect to some basis of V if and only if

  $$\dim V = \dim \text{null}(T - \lambda_1 \text{id}_V) + \dots + \text{null}(T - \lambda_k \text{id}_V),$$

  where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of T.
- 0 is an eigenvalue of $T \in L(V)$ if and only if $\text{null}(T) \neq \{0\}$.