Math 110, Fall 2013. Matrices Review

Let V, W be F-vector spaces and $T \in L(V, W)$. Suppose that $B = (v_1, \dots, v_n) \subset V$ and $C = (w_1, \dots, w_m) \subset W$ are bases. Then, there exists a unique $m \times n$ matrix, denoted $[T]_B^C$, such that

$$[T(v)]_C = [T]_B^C [v]_B, \quad \text{for every } v \in V.$$

Recall that $[v]_B$ is the B-coordinate vector of v - since B is a basis we can find unique $a_1, \dots, a_n \in F$ such that $v = a_1 v_1 + \dots + a_n v_n$. Then,

$$[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$$

Similarly, since C is a basis we know that we can write $T(v) \in W$ as a linear combination $T(v) = b_1 w_1 + \dots + b_m w_m$. Then,

$$[T(v)]_C = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in F^m$$

The matrix $[T]_B^C$ is the matrix

$$[T]_B^C = [[T(v_1)]_C [T(v_2)]_C \cdots [T(v_n)]_C]$$

so its i^{th} column is the C-coordinate vector of $T(v_i)$.

For example, consider the linear map

$$T : P_2(\mathbb{R}) \to Mat_{2\times 2}(\mathbb{R}) \; ; \; p \mapsto \begin{bmatrix} p(1) & p(2) \\ 0 & p(-1) \end{bmatrix}$$

Let $B = (1, z, z^2)$ and $C = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$. Then, we find that

$$T(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow [T(1)]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$T(z) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \Rightarrow [T(z)]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix},$$

$$T(z^2) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \Rightarrow [T(z^2)]_C = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Hence, we have

$$[T]_B^C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

It is also possible to gain information about subspaces of V by looking at the matrix of an operator $T \in L(V)$ with respect to some basis. For example, if $B = (v_1, \dots, v_n)$ is a basis and

$$[T]_B = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix and C is a $(n-k) \times (n-k)$ matrix, then we have that $T(v_i) \in \text{span}(v_1, \dots, v_k)$, for $i = 1, \dots, k$. Hence, $\text{span}(v_1, \dots, v_k)$ is a T-invariant subspace. Note that the '0' in the bottom left is an $(n-k) \times k$ matrix containing all 0's.

Conversely, if U is T-invariant and if we choose $(u_1, \dots, u_k)$ is a basis of U, then we can extend to a basis $B = (u_1, \dots, u_k, v_1, \dots, v_l)$. Then, the matrix of T with respect to B looks like

$$[T]_B = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

as above.