Math 110, Fall 2013. Jordan Form Review

Make sure that you know how to compute the matrix of an operator $T \in L(V)$ with respect to a basis. Also, make sure you know the information contained within such a matrix - eg. if the first column of $[T]_B$ (for a basis $B = (v_1, \dots, v_n)$ of V) is e_1 then you know that $T(v_1) = v_1$.

First we provide the Theorem that we've been building up to

Theorem (Jordan Form) Let V be a complex, finite dimensional vector space, $T \in L(V)$ be an operator whose distinct eigenvalue are $\lambda_1, \dots, \lambda_k$ (ie, each eigenvalue of T appears exactly once in this list). Then, there exists a basis $B = (v_1, \dots, v_n)$ of V such that

$$[T]_B = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots \ddots & \vdots & \\ 0 & \cdots & \cdots & A_k \end{bmatrix},$$

where each A_i is square matrix of size $\dim \tilde{E}_{\lambda_i}$ - recall that $\tilde{E}_{\lambda_i}$ is the generalised λ_i-eigenspace - and, moreover, we can further decompose each A_i as

$$A_i = \begin{bmatrix} J(\lambda_i, r_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_i, r_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & J(\lambda_i, r_{s_i}) \end{bmatrix}$$

where each $J(\lambda_i, r_j)$ is a λ_i-Jordan block - a square $r_j \times r_j$ matrix of the form

$$J(\lambda_i, r_j) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_i & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda_i \end{bmatrix}$$

ie, '**λ_i's on the diagonal, 1's above the diagonal and zero elsewhere**'. We usually order the λ_i-Jordan blocks so that $r_1 \geq r_2 \geq \cdots \geq r_{s_i} \geq 1$.

We say that $[T]_B$ is in **Jordan form**, and that B is a **Jordan basis**.

For example, the following matrices are in Jordan form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

while the following matrices are not in Jordan form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Given the Jordan form of an operator we can determine the **characteristic polynomial** χ_T and the **minimal polynomial** m_T as follows: we have

$$\chi_T(z) = \prod_{i=1}^{k}(z - \lambda_i)^{d_i}, \quad m_T(z) = \prod_{i=1}^{k}(z - \lambda_i)^{e_i}$$

where

$$d_i = \text{ no. of times } \lambda_i \text{ appears on the diagonal of the Jordan form of } T,$$

$$e_i = \text{ the size of the largest } \lambda_i\text{-Jordan block.}$$

The statement about the e_i requires a little justification - that is, we need to prove that this is indeed the case - but it's best to look at an example. Suppose that $T \in L(\mathbb{C}^5)$ is an operator and we've found a Jordan basis so that the Jordan form of T is

$$[T]_B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Thus, the characteristic polynomial is

$$\chi_T(z) = (z-1)^3(z-2)^2.$$

We claim that the minimal polynomial is

$$p(z) = (z-1)^2(z-2)^2.$$

Recall that the minimal polynomial m_T is the polynomial of smallest degree and with leading coefficient 1 (ie $m_T(z) = z^r + \sum_{i=0}^{r-1} a_i z^i$) such that $m_T(T) = 0 \in L(V)$.

Denote the basis $B = (v_1, \dots, v_5)$, so that the Jordan form matrix tells us that

$$T(v_1) = v_1,\ T(v_2) = v_2 + v_1,\ T(v_3) = v_3,\ T(v_4) = 2v_4,\ T(v_5) = 2v_5 + v_4.$$

Then, you can check that the above equations imply that

$$(T-1)(v_1) = (T-1)(v_3) = (T-1)^2(v_2 = 0,$$

$$(T-2)(v_4) = (T-2)^2(v_5),$$

and that $(T-1)(v_2) \neq 0$, $(T-2)(v_5) \neq 0$. Thus, to show that

$$p(T) = (T-1)^2(T-2)^2 = 0 \in L(\mathbb{C}^5),$$

it suffices to show that $p(T)(v_i) = 0 \in \mathbb{C}^5$, for each $i = 1, \dots, 5$.

Indeed, noting that $(T-1)^2(T-2)^2 = (T-2)^2(T-1)^2$ in $L(\mathbb{C}^5)$, we have

$$p(T)(v_1) = (T-2)^2(T-1)^2(v_1) = (T-2)^2(T-1)(0) = 0,$$

$$p(T)(v_2) = (T-2)^2(T-1)^2(v_2) = (T-2)^2(0) = 0,$$

$$p(T)(v_3) = (T-2)^2(T-1)^2(v_3) = (T-2)^2(T-1)(0) = 0,$$

and similarly for v_4, v_5. Hence, we have that $p(T) = 0 \in L(\mathbb{C}^5)$.

But how to show that this is the minimal polynomial? Well, recall the following facts:

- **if $p \in P(\mathbb{C})$ is a polynomial such that $p(T) = 0 \in L(V)$, then m_T divides p,**

- **the roots of m_T are precisely the eigenvalues of T**

Hence, we know that m_T divides p above and that $m_T = (z-1)^{e_1}(z-2)^{e_2}$. Since we have that

$$(T-1)(v_2) \neq 0, \quad \text{and} \quad (T-2)(v_5) \neq 0,$$

then it is not possible that $(T-1)(T-2)^2 = 0 \in L(\mathbb{C}^5)$ or $(T-1)^2(T-2) = 0 \in L(\mathbb{C}^5)$ - here we are also using that $\tilde{E}_1 \cap \tilde{E}_2 = \{0\}$.

Hence, we must have that m_T divides $p = (z-1)^2(z-2)^2$ and its roots are $1, 2$, and we also have that $(T-1)(T-2)^2 \neq 0$, $(T-1)^2(T-2) \neq 0$, so that $m_T = p$.

What have we really shown here? We've shown (this requires a bit of thought) that if $v \in \tilde{E}_\lambda$ is a generalised eigenvector such that $(T-\lambda)^j(v) = 0$, while $(T-\lambda)^{j-1}(v) \neq 0$, then the Jordan form of T must contain a λ-Jordan block whose size is at least $j \times j$. Hence, we see that if

$$\dim \operatorname{null}(T-\lambda)^j > \dim \operatorname{null}(T-\lambda)^{j-1},$$

then the Jordan form of T contains a λ-Jordan block whose size is at least $j \times j$.

In fact, we have the following

$$\dim \operatorname{null}(T-\lambda)^j - \dim \operatorname{null}(T-\lambda)^{j-1} = \text{ no. of } \lambda\text{-Jordan blocks of size at least } j \times j.$$

In particular,

$$\dim \operatorname{null}(T-\lambda) = \text{ no. of } \lambda\text{-Jordan blocks of size at least } 1 \times 1 = \text{no. of } \lambda\text{-Jordan blocks.}$$

Let's see how we can use this information to help us: consider the operator

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x}$$

determine the Jordan form of T.

Since the matrix of T with respect to the standard basis $S = (e_1, \dots, e_4)$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is upper triangular, we see that T admits precisely one eigenvale $\lambda = 0$. So, the Jordan form for T must be one of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

as these are all possible ways we can put 0-Jordan blocks of size no larger than 4×4 into a 4×4 matrix, and such that we put the largest blocks in the top left.

So, the Jordan form of T is one of these matrices, but which one? We can obviously exclude the zero matrix as T is not the zero operator (eg $T(e_2) = e_1 \neq 0$). So we are left with four possibilities. Now, we've seen above that

$$\dim \operatorname{null}(T) = \text{ no. of 0-Jordan blocks}$$

so we compute $\dim \operatorname{null}(T)$: this is the number of free variables/non-pivot columns in the matrix A (this is Math 54 stuff), and it's straightforward to see that there are two non-pivot columns. Hence, we must have $\dim \operatorname{null}(T) = 2$, so that the Jordan form has two 0-Jordan blocks. Thus, the only such possibilities are the third and fourth matrices on the list above.

So, how can we distinguish between these two? Notice that if T had Jordan form

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some basis $B = (v_1, \dots, v_4)$ then we have

$$T(v_1) = 0, \ T(v_2) = v_1, \ T(v_3) = v_2,$$

so that

$$T^3(v_3) = 0, \ \text{ and } \ T^2(v_2) \neq 0.$$

Hence, we have $\operatorname{null} T^3 \neq \operatorname{null} T^2$. Whereas, if T had Jordan form

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then we have $\operatorname{null} T^2 = \mathbb{C}^4$. In particular, $\operatorname{null} T^3 = \operatorname{null} T^2$. Hence, we can distinguish between these two Jordan forms by determining the dimensions of $\operatorname{null} T^2$ - if

$$\dim \operatorname{null} T^2 = \dim \mathbb{C}^4 = 4,$$

then we have two 2×2 Jordan blocks; if

$$\dim \operatorname{null} T^2 < 4$$

then we have one 3×3 Jordan block and one 1×1 Jordan block. Now,

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\dim \operatorname{null} T^2 = 3$, and we must have that the Jordan form of T is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercises: Determine the Jordan form of the following operators

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \ ; \ \underline{x} \mapsto A\underline{x},$$

where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Finding the Jordan basis

I will now present an algorithm for determining the Jordan basis of an operator $T \in L(V)$ with exactly one eigenvalue λ.

1) Determine the eigenvalue of T (eg, entries on diagonal of upper/lower triangular matrix defining T)

2) Determine the smallest integer e such that $\text{null}(T - \lambda)^e = V$ - this means take any basis B of V (ie, the standard basis if $V = \mathbb{C}^4$) and determine the matrix $C = [T]_B - \lambda I_n$. Then, compute C^j until $C^e = 0$ is the zero matrix. This is the value of e you are looking for.

3) Denote

$$H_e = V, \ H_{e-1} = \text{null}(T - \lambda)^{e-1}, \ \dots \ , H_1 = \text{null}(T - \lambda), \ H_0 = \{0\}$$

and compute bases for each of $H_1, \dots, H_e$.

4) Determine $G_{e-1} \subset V$ a subspace such that

$$H_e = G_{e-1} \oplus H_{e-1},$$

and let $(v_1^{(e)}, \dots, v_{r_e}^{(e)})$ be a basis of G_{e-1}.

5) Determine G_{e-2} such that

$$H_{e-1} = H_{e-2} \oplus \text{span}(T(v_1^{(e)}), \dots, T(v_{r_e}^{(e)})) \oplus G_{e-2}$$

and let $(v_1^{(e-1)}, \dots, v_{r_{e-1}}^{(e-1)})$ be a basis of G_{e-2}.

6) Determine G_{e-3} such that

$$H_{e-2} = H_{e-3} \oplus \text{span}(T^2(v_1^{(e)}), \dots, T^2(v_{r_e}^{(e)}), T(v_1^{(e-1)}), \dots, T(v_{r_{e-1}}^{(e-1)})) \oplus G_{e-3}$$

$\vdots$

?) Determine G_0 such that

$$H_1 = \text{span}(T^{e-1}(v_1^{(e)}), \dots, T^{e-1}(v_{r_e}^{(e)}), T^{e-2}(v_1^{(e-1)}), \dots, T^{e-2}(v_{r_{e-1}}^{(e-1)}), \dots, T(v_1^{(2)}), \dots, T(v_{r_2}^{(2)})) \oplus G_0$$

and let $(v_1^{(1)}, \dots, v_{r_1}^{(1)})$ be a basis of G_0.

END! Write the vectors we've obtained (all the v's) in the following table

$$\begin{array}{ccccccccccc}
v_1^{(e)} & v_2^{(e)} & \cdots & v_{r_e}^{(e)} & & & & & & & \\
T(v_1^{(e)}) & T(v_2^{(e)}) & \cdots & T(v_{r_e}^{(e)}) & v_1^{(e-1)} & \cdots & v_{r_{e-1}}^{(e-1)} & & & & \\
T^2(v_1^{(e)}) & T^2(v_2^{(e)}) & \cdots & T^2(v_{r_e}^{(e)}) & T(v_1^{(e-1)}) & \cdots & T(v_{r_{e-1}}^{(e-1)}) & \cdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & & \\
T^{e-1}(v_1^{(e)}) & T^{e-1}(v_2^{(e)}) & \cdots & T^{e-1}(v_{r_e}^{(e)}) & T^{e-2}(v_1^{(e-1)} & \cdots & T^{e-1}(v_{r_{e-1}}^{(e-1)}) & \cdots & v_1^{(1)} & \cdots v_{r_1}^{(1)}
\end{array}$$

and list them in following order: start at the bottom of the leftmost column and list the vectorsas you go up the first column, once you get to the top go to the bottom of the second column and list the vectors as you go up this column, once you get to the top go to the bottom of the third column and list the vectors as you go up... etc. The resulting list is a Jordan basis(!).

As is evident from the algorithm, this is not such an easy process and can be very confusing and easy to make a mistake. Let's see an example: consider the operator T from the previous worked example. We saw that the Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let's determine a Jordan basis B such that

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) We know that the eigenvalue os $\lambda = 0$.

2) We have $e = 3$ (we've seen this above, as we already know the Jordan form)

3) Define

$$H_1 = \text{null}(T) = \text{span}(e_1, -e_2 - e_3 + e_4),\ H_2 = \text{null}(T^2) = \text{span}(e_1, e_2, -e_3 + e_4),\ H_3 = \mathbb{C}^4$$

4) Set $G_2 = \text{span}(e_4)$. Then, $(e_1, e_2, -e_3 + e_4, e_4)$ is linearly independent, hence a basis of$\mathbb{C}^4$, so that we have

$$\mathbb{C}^4 = H_2 \oplus G_2.$$

5) Now, $T(e_4) = e_1 + e_2$, and $\dim \text{span}(e_1 + e_2) \oplus H_1 = 3 = \dim H_2$ so that we take $G_1 = \{0\}$.

6) We have $T^2(e_4) = T(e_1 + e_2) = e_1$, and $\dim \text{span}(e_1) \oplus H_0 = 1$, while $\dim H_1 = 2$. Hence, we can find a G_0 - since $H_1 = \text{span}(e_1, -e_2 - e_3 + e_4)$ we can take $G_0 = \text{span}(-e_2 - e_3 + e_4)$.

7) We write the above vectors in the table

$$\begin{array}{cc}
e_4 & \\
e_1 + e_2 & \\
e_1 & -e_2 - e_3 + e_4
\end{array}$$

and order them as

$$B = (v_1, v_2, v_3, v_4) = (e_1, e_1 + e_2, e_4, -e_2 - e_3 + e_4)$$

Then, we have

$$T(v_1) = 0, \ T(v_2) = T(e_1 + e_2) = e_1 = v_1,$$

$$T(v_3) = e_1 + e_2 = v_2, \ T(-e_2 - e_3 + e_4) = -e_1 - e_2 + e_1 + e_2 = 0.$$

Hence, we have

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise: Find Jordan bases for each of the operators given in the previous exercise.

Remark: notice that the table we wrote down above has two (= no. of Jordan blocks) columns and that the height of the columns corresponds to the size of the Jordan blocks that appear in the Jordan form (ie, the first column has height 3, the second has height 1). This peculiarity will hold always - the number of columns in your table equals the number of Jordan blocks; each column corresponds to a Jordan block whose size is equal to the height of the column.

Note also that the vectors in the bottom i rows form a basis of $\text{null}\, T^i$.

Let's try another example: consider the operator

$$T : \mathbb{C}^3 \to \mathbb{C}^3 \ ; \ \underline{x} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}$$

What are the eigenvalues of T? We notice that $T^3 = 0 \in L(\mathbb{C}^3)$ since the matrix defining T, let's call it A, satisfies $A^3 = 0$, while $A^2 \neq 0$. Hence, the minimal polynomial of T divides z^3. Since T is not the zero operator - so that $m_T \neq z$ - and $A^2 \neq 0$, we must have $m_T = z^3$. Hence, by our above discussion, we know that the size of the largest Jordan block in the Jordan form of T is 3×3. There is only one possibility for a 3×3 matrix being in Jordan form and containing only 0-Jordan blocks, one of which is 3×3. Namely,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, we know the Jordan form of T. What about a Jordan basis?

Denote

$$H_3 = \mathbb{C}^3, \ H_2 = \text{null}(T^2), \ H_1 = \text{null}(T), \ H_0 = \{0\}$$

We compute that

$$H_1 = \text{null}(T) = \text{null}(A) = \text{span}(e_1), \ H_2 = \text{null}(T^2) = \text{null}(A^2) = \text{span}(e_1, e_3).$$

So, we can take $G_2 = \text{span}(e_2)$ so that

$$H_3 = G_2 \oplus H_2.$$

Since we know there is onnly one Jordan block in our Jordan form, we know there will be exactly one column in the table we create above (see the Remark) - namely, we'll get

$$\begin{matrix} e_2 \\ e_3 \\ e_1 \end{matrix}$$

where we have used that $T(e_2) = e_3$, $T^2(e_2) = e_1$. Hence, a Jordan basis is

$$B = (v_1, v_2, v_3) = (e_2, e_3, e_1),$$

and

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$