Math 110, Fall 2013. Jordan Form Review

First and foremost:

- make sure that you know how to determine the matrix $[T]_B$ of an operator $T \in L(V)$ with respect to a basis B.
- make sure that you know how to find bases of null spaces of operators/matrices (this is Math 54 stuff - ie, row-reduction).

Things you must know:

Ch. 6 - inner products, inner product spaces, norm, orthogonality, Euclidean inner product, inner product space = geometry, orthonormal bases, Gram-Schmidt, orthogonal complement, orthogonal projection, functionals, adjoints.

Ch. 7 - self-adjoint operators, normal operators, Spectral Theorems (real/complex), normal operators on real inner product spaces.

Ch. 8 - generalised eigenvalues, nilpotent operators, characteristic polynomial, minimal polynomial, Jordan form, Jordan basis.

Remark: since the final exam is cumulative you are also expected to 'know' all of the previous material(!). Of course, you should spend the majority of your time going over the material from Ch. 6-8, but you should definitely not have forgotten about things like *linear (in)dependence, spans, bases, T-invariant, eigenstuff, etc*.

Theorems

Ch. 6

- Pythagoras' Theorem, Cauchy-Schwartz, triangle inequalities, parallelogram equality.
- orthogonal lists are linearly independent (but not conversely!).
- Gram-Schmidt process.
- orthonormal bases ALWAYS exist for an inner product space (in particular, if you have an inner product space, you should be thinking '*choose an orthonormal basis $B = (v_1, \dots, v_n)$ of V*').
- orthonormal lists can be extended to orthonormal bases (note, you must start with an orthonormal list!).
- if T admits an upper-triangular matrix with respect to some basis, then it admits an upper-triangular matrix with respect to some orthonormal basis.
- $V = U \oplus U^{\perp}$ for **any** subspace U of V.
- $(U^{\perp})^{\perp} = U$. In words '**the complement of the complement is what you started with**'.
- the orthogonal projection of v onto U is the vector in U that is closest to v (Prop. 6.36). In particular, if the orthogonal projection of v onto U is v, then $v \in U$; if the orthogonal projection of v onto U is 0 then $v \in U^{\perp}$.
- if $(v_1, \dots, v_n)$ is an orthonormal basis, and $v \in V$, then

$$v = \langle v, v_1 \rangle v_1 + \dots + \langle v, v_n \rangle v_n$$

- if $v \in V$ is such that $\langle u, v\langle = 0$, for every $u \in V$, then $v = 0$.
- every linear functional $\alpha \in L(V, F)$ is of the form $\alpha = \langle -, v\rangle$, for some **unique** $v \in V$. That is, if $\alpha \in L(V, F)$, then there is a **unique** $v \in V$ such that, $\alpha(u) = \langle u, v\rangle$, for every $u \in V$.
- properties of the adjoint (p. 119)
- Proposition 6.46 (know how to replicate the proof given)
- if $T \in L(V, W)$ and $B \subset V$, $C \subset W$ are **orthonormal** bases, then

$$[T^*]_C^B = \overline{[T]_B^C}^t, \quad (\textit{conjugate transpose})$$

Ch. 7

- eigenvalues of self-adjoint operators are real.
- $T \in L(V)$ is normal if and only if $||Tv|| = ||T^*v||$, for every $v \in V$.
- if T is normal then $\text{null}\, T = \text{null}\, T^*$.
- if T is normal then $\text{range}\, T = \text{range}\, T^*$.
- if T is normal then $\text{null}\, T \cap \text{range}(T) = \{0\}$ (since $\text{range}(T) = \text{null}(T^*)^\perp$).
- if T is normal and $v \in V$ is an eigenvector so that $T(v) = \lambda v$, then $T^*(v) = \overline{\lambda} v$.
- if $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of T, where T is normal, and $v_1, \dots, v_k$ are corresponding eigenvectors, then $\langle v_i, v_j\rangle = 0$, for $i \neq j$. In words: '**eigenvectors of normal operators associated to distinct eigenvalues are orthogonal**'.
- (Complex Spectral Theorem) if V is complex, $T \in L(V)$. Then, T is normal if and only if V admits an orthonormal basis of eigenvectors of T. This implies that, if T is normal then there is an orthonormal basis B of V such that $[T]_B$ is a diagonal matrix.
- Let T be normal operator on complex vector space. Then, T is self-adjoint if and only if all of its eigenvalues are real.
- (Real Spectral Theorem) if V is real, $T \in L(V)$. Then, T is self-adjoint if and only if V admits an orthonormal basis of eigenvectors of T. This implies that, if T is self-adjoint then there is an orthonormal basis B of V such that $[T]_B$ is a diagonal matrix.

Ch. 8

- let $\dim V = n$. Then, $\tilde{E}_\lambda = \text{null}(T - \lambda)^n$.
- if $\text{null}(T - \lambda)^j = \text{null}(T - \lambda)^{j+1}$, then $\text{null}(T - \lambda)^j = \text{null}(T - \lambda)^i$, for every $i \geq j$.
- if N is nilpotent then $N^{\dim V} = 0 \in L(V)$.
- let T be an operator whose distinct eigenvalues are $\lambda_1, \dots, \lambda_k$. The characteristic polynomial of T is

$$\chi_T = \prod_{i=1}^{k} (z - \lambda_i)^{d_i},$$

  where $d_i = \dim \tilde{E}_{\lambda_i}$.
- d_i is the number of times that λ_i appears on the diagonal of an upper-triangular matrix representing T.
- χ_T has degree $\dim V$. Hence, $d_1 + \dots + d_k = \dim V$.

- (Cayley-Hamilton) we have

$$\chi_T(T) = \prod_{i=1}^{k}(T - \lambda_i)^{d_i} = 0 \in L(V).$$

- there is a basis of V consisting of generalised eigenvectors of T. In particularm there exists a basis B of V such that

$$[T]_B \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}$$

  where each A_i is a square matrix of size $d_i = \dim \tilde{E}_{\lambda_i}$.

- generalised eigenspace are T-invariant.
- the minimal polynomial of T is

$$m_T = \prod_{i=1}^{k}(z - \lambda_i)^{e_i},$$

  where $1 \leq e_i \leq d_i$, for each i.

- e_i is the smallest integer such that $\tilde{E}_{\lambda_i} = \text{null}(T - \lambda_i)^{e_i}$.
- if $p \in P(\mathbb{C})$ is a polynomial such that $p(T) = 0 \in L(V)$. Then, m_T divides p. In particular, if $p(T) = 0 \in L(V)$, then the eigenvalues of T are a subset of the roots of p.
- Jordan form (see the note on the Jordan form on my website).