HW3. Math 110, Fall 2013. Additional Problem Solution

Let V, W be vector spaces (over F) and U_1, U_2 subspaces such that $V = U_1 \oplus U_2$. Denote

$$L = L(V, W),\ L_1 = L(U_1, W),\ L_2 = L(U_2, W).$$

We want to find subspaces of L, K_1 and K_2 say, such that L_i is isomorphic to K_i, and $L = K_1 \oplus K_2$.

We take

$$K_1 = \{T \in L \mid T(v) = 0_W,\ \text{for any } v \in U_2\},$$
$$K_2 = \{T \in L \mid T(v) = 0_W,\ \text{for any } v \in U_1\}.$$

These are both subspaces of L: we show that K_1 is a subspace, the proof being (essentially) the same for K_2. So, let $T, S \in K_1, \lambda\mu \in F$, we want to show that $\lambda T + \mu S \in K_1$. Now, if $v \in U_2$, then

$$(\lambda T + \mu S)(v) = \lambda T(v) + \mu S(v) = \lambda 0_W + \mu 0_W = 0_W$$

Hence, $\lambda T + \mu S \in K_1$.

Let $T \in L$. As $V = U_1 \oplus U_2$, we can write any $v \in V$ uniquely as $v = u_1 + u_2$, for some (unique!) $u_1 \in U_1, u_2 \in U_2$. Define $T_1, T_2 \in L$ as follows: for $v = u_1 + u_2 \in V$,

$$T_1(v) = T(u_1),\ T_2(v) = T(u_2).$$

This definition is well-defined since there is only one choice $u_1 \in U_1, u_2 \in U_2$ for which $v = u_1 + u_2$. It is straightforward to check that $T_1, T_2 \in L$ (ie, they are linear maps). Moreover, if $v \in U_2$ then $v = 0 + v \in U_1 + U_2$ is its unique decomposition into a sum of elements from U_1 and U_2 and $T_1(v) = 0$. Hence, $T_1 \in K_1$. Similarly, we find that $T_2 \in K_2$. Hence, we have shown that $L = K_1 + K_2$. Now, suppose that $T \in K_1 \cap K_2$. Then, for any $v \in U_1$, we have $T(v) = 0$ (since $T \in K_2$) and for any $v \in U_2$ we have $T(v) = 0$ (since $T \in K_1$). In particular, if $v = u_1 + u_2$, $u_1 \in U_1, u_2 \in U_2$, then $T(v) = T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + 0 = 0$. So, T is the zero linear map and $K_1 \cap K_2 = \{0\}$. Hence, $L = K_1 \oplus K_2$.

Now, we show that L_i is isomorphic to K_i, for $i = 1, 2$: thus, we must describe an invertible linear map

$$f_i : L_i \to K_i,\quad \text{for } i = 1, 2.$$

Define f_1 as follows: to any $S \in L_1$ we can extend S to a linear map $\tilde{S} \in L$, so that $\tilde{S}(v) = 0$, for any $v \in U_2$ (ie, choose a basis $(b_1, \dots, b_m)$ of U_1 and extend to a basis $(b_1, \dots, b_m, b_{m+1}, \dots, b_n)$ of V and define $\tilde{S}(b_i) = S(b_i)$, for $i = 1, \dots, m$, and $\tilde{S}(b_i) = 0$, for $i > m$). Hence, $\tilde{S} \in K_1$. We can do a similar extension for any $R \in L_2$ to obtain a linear map $\tilde{R} \in L$ such that $\tilde{R}(v) = 0$, for any $v \in U_1$. We now define

$$f_1(S) = \tilde{S},\ f_2(R) = \tilde{R},\ S \in L_1, R \in L_2$$

We have to show that f_i is linear and invertible.

Let $S, S' \in L_1$ and denote $Z = S + S' \in L_1$. Then, $\tilde{Z} \in K_1$ is the linear map such that, for $u_1 \in U_1, u_2 \in U_2$,

$$\tilde{Z}(u_1 + u_2) = \tilde{Z}(u_1) + \tilde{Z}(u_2) = (S + S')(u_1) + 0 = S(u_1) + S'(u_1)$$

We also have

$$(\tilde{S} + \tilde{S}')(u_1 + u_2) = \tilde{S}(u_1 + u_2) + \tilde{S}'(u_1 + u_2) = S(u_1) + S'(u_1)$$

so that $\tilde{Z} = \tilde{S} + \tilde{S}'$. That is, $f_1(S+S') = f_1(S) + f_1(S')$. We can also show that $f_1(cS) = cf_1(S)$ by similar considerations. Hence, f_1 is linear. In an analagous way we can show that f_2 is linear.

we now show that f_i are invertible, considering the case of f_1 first:

f_1 injective: let $S \in L_1$ and suppose that $f_1(S) = 0 \in L$ is the zero linear map. Thus, for any $v \in U_1$ we have

$$0 = f_1(S)(v) = \tilde{S}(v) = S(v) \implies S = 0 \in L_1.$$

Similarly, we can show that f_2 is injective.

f_2 is surjective: let $T \in K_1$, we want to find $S \in L_1$ such that $f_1(S) = T$, ie, $\tilde{S} = T$. Define, for any $v \in U_1$, $S(v) = T(v)$. Then, $S \in L_1$ (ie S is linear) and

$$\tilde{S}(u_1 + u_2) = S(u_1) = T(u_1) = T(u_1 + u_2), \text{ since } T \in K_1.$$

Hence, $\tilde{S} = T$. In a similar way we can show that f_2 is surjective. Hence, f_i are linear and invertible, therefore they are isomorphisms.