Worksheet 11/20. Math 110, Fall 2013. SOLUTIONS

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Normal and Self-Adjoint Operators, Spectral Theorem

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.

1. a) Give an example of an operator $T \in L(\mathbb{C}^2)$ that is not a normal operator. Explain carefully why you know it is not a normal operator.

b) Give an example of a diagonalisable operator $T \in L(\mathbb{C}^2)$ that is not normal. Justify your chosen example carefully.

c) Give an example of an operator $T \in L(\mathbb{R}^2)$ that is diagonalisable but not self-adjoint.

Solution: a) For example, the operator

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not normal as it is not diagonalisable (contradicting the Complex Spectral Theorem). It isn't diagonalisable since there is precisely one eigenvalue ($\lambda = 0$, the diagonal entries of the matrix defining T) and in order for T to be diagonalisable we would require that there are two linearly independent eigenvectors associated with this eigenvalue. However, it is straightforward to check the the 0-eigenspace is span(e_1), which has dimension one, so that it is impossible to find two linearly independent eigenvectors.

b) Consider the operator $T \in L(\mathbb{C}^2)$ defined on the basis $B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$: say, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This operator is diagonalisable ($\mathbb{C}^2$ admits a basis of eigenvectors B. However, T is not normal - if it were normal then we would require eigenvectors associated to distinct eigenvalues to be orthogonal (wrt to the Euclidean inner product, which is what we are assuming since no other inner product was specified). However, it is easy to check that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.1 + 0.1 = 1 \neq 0,$$

so that the eigenvectors of T are not orthogonal.

c) We can use the same example from part b) - if T were to be self-adjoint then the eigenvectors would have to form an orthogonal basis of $\mathbb{R}^2$. As we've just seen, this is not the case, despite T being diagonalisable.

2. Let $(\mathbb{R}^2, \langle , \rangle)$ be the inner product space, with

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2, \; \underline{x}, \underline{y} \in \mathbb{R}^2.$$

a) Define a self-adjoint operator T on the inner product space $(\mathbb{R}^2, \langle , \rangle)$ that has eigenvalues $\sqrt{2}, 1$.

b) Is it possible for an operator on this inner product space to have exactly one eigenvalue? If so, can you give an example? If not, can you prove it?

c) Is the linear operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x},$$

a self-adjoint operator on the inner product space $(\mathbb{R}^2, \langle , \rangle)$?

Solution: a) In order to define a self-adjoint operator we first want to find an orthogonal basis of $\mathbb{R}^2$ with respect to $\langle , \rangle$. So, we perform the Gram-Schmidt process on the basis (e_1, e_2) thereby producing an orthonormal (relative to the given inner product) basis (v_1, v_2) of $\mathbb{R}^2$. So, set

$$v_1 = \frac{e_1}{||e_1||} = \frac{1}{\sqrt{\langle e_1, e_2 \rangle}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_2' = e_2 - \langle e_2, v_1 \rangle v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - (-\frac{1}{\sqrt{2}}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \implies v_1 = \frac{v_2'}{||v_2'||} = \sqrt{2} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Now, define T on the basis (v_1, v_2): set

$$T(v_1) = \sqrt{2} v_1, \;\; T(v_2) = v_2$$

Then, T is self-adjoint - it admits an orthonormal basis of eigenvectors of T, so the Spectral Theorem implies the result - and its eigenvalues are $\sqrt{2}, 1$, by conbstruction.

b) Yes, take the zero operator.

c) If it were self-adjoint then its eigenvectors would need to be orthogonal relative to $\langle , \rangle$. The eigenvalues of T are $\lambda = 0, 2$ - obtained by determining those λ for which $x_1 + x_2 = \lambda x_1, \; x_1 + x_2 = \lambda x_2$, admits a nonzero solution (or use characteristic polynomial) - and the corresponding eigenspaces are span $\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ and span $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$. Now, we have

$$\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle = 2 - 1 + 1 - 1 = 1 \neq 0,$$

so that T is not self-adjoint.

3. Say that an $n \times n$ matrix Q with real entries is *orthogonal* if its columns form an orthonormal basis of $\mathbb{R}^n$ with the Euclidean inner product; an $n \times n$ matrix Q with complex entries is *unitary* if it satisfies the analogous condition. Prove that the following properties of a square matrix over $\mathbb{R}$ or $\mathbb{C}$ are equivalent:

(a) Q is unitary (if $F = \mathbb{C}$) or orthogonal (if $F = \mathbb{R}$).

(b) QQ^* is the identity matrix.

(c) The conjugate transpose Q^* is unitary (if $F = \mathbb{C}$) or orthogonal (if $F = \mathbb{R}$).

(d) The rows of Q form an orthonormal basis of F^n $(F = \mathbb{R}, \mathbb{C})$.

Here we are using the notation $Q^* = \overline{Q}^t$. (*Hint: you need to show that (a)* $\implies$ *(b)* $\implies$ *(c)* $\implies$ *(d)* $\implies$ *(a).*)

Solution: (a) $\Rightarrow$ (b) Denote the columns of Q by $a_1, \dots, a_n$. Then, consider the product $Q^* Q$. We see that the top row of Q^* is $(\overline{a}_1)^t$, ie, the row vecto whose i^{th} entry is the conjugate of the i^{th} entry of a_i. Now, consider how matrix multiplication works - to obtain the entry in the first row and j^{th} column, we take the first row of Q^* and we 'dot' it with the j^{th} column of Q. That is, we compute

$$\langle a_j, a_1 \rangle$$

Hence, we have seen that the $1j$ entry of Q^*Q is $\langle a_j, a_1 \rangle$. Now, the same reasoning shows that the ij-entry of Q^*Q is $\langle a_j, a_i \rangle$. Hence, since the columns of Q are orthonormal (by assumption) we see that

$$Q^*Q = I_n$$

Hence, Q^* is the inverse matrix of Q, so that we must automatically have $QQ^* = I_n$ (if A, B are square and $AB = I_n$ then $BA = I_n$).

(b) $\Rightarrow$ () We have seen that $QQ^* = I_n$ if and only if $Q^*Q = I_n$. To show that Q^* is unitary we must show that its columns are orthonormal. The columns of Q^* are the conjugates of the rows of Q. Denote the rows of Q by $b_1, \dots, b_n$. Thus, by considering how matrix multiplication is defined for QQ, we see that the ij-entry of this product is $\langle b_i, b_j \rangle$. Hence, by assuming that $QQ^* = I_n$ we are stating that $\langle \overline{b}_j, \overline{b}_i \rangle = \langle b_i, b_j \rangle = 0$, when $i \neq j$. and $\langle \overline{b}_i, \overline{b}_i \rangle = \langle b_i, b_i \rangle = 1$. This means that the conjugates of the rows of Q form an orthonormal basis of F^n - therefore, the columns of Q^* form a basis of F^n.

(c) $\Rightarrow$ (d) This follows from the last couple of lines in the previous argument.

(d) $\Rightarrow$ (a) If the rows of Q form an orthonormal basis of F^n then we must have $QQ^* = I_n$ - the equations stating orthonormality of rows is precisely captured by this matrix equation. We need to show that the columns of Q are also orthonormal. We have seen that $QQ^* = I_n$ if and only if $Q^*Q = I_n$. Now, as already mentioned above, this matrix equation captures the statement that the columns of Q are orthonormal. Hence, Q is unitary.