Worksheet 11/20. Math 110, Fall 2013.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Normal and Self-Adjoint Operators, Spectral Theorem

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.

1. a) Give an example of an operator $T \in L(\mathbb{C}^2)$ that is not a normal operator. Explain carefully why you know it is not a normal operator.

b) Give an example of a diagonalisable operator $T \in L(\mathbb{C}^2)$ that is not normal. Justify your chosen example carefully.

c) Give an example of an operator $T \in L(\mathbb{R}^2)$ that is diagonalisable but not self-adjoint.

2. Let $(\mathbb{R}^2, \langle , \rangle)$ be the inner product space, with

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2, \ \underline{x}, \underline{y} \in \mathbb{R}^2.$$

a) Define a self-adjoint operator T on the inner product space $(\mathbb{R}^2, \langle , \rangle)$ that has eigenvalues $\sqrt{2}, 1$.

b) Is it possible for an operator on this inner product space to have exactly one eigenvalue? If so, can you give an example? If not, can you prove it?

c) Is the linear operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \ ; \ \underline{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x},$$

a self-adjoint operator on the inner product space $(\mathbb{R}^2, \langle , \rangle)$?

3. Say that an $n \times n$ matrix Q with real entries is *orthogonal* if its columns form an orthonormal basis of $\mathbb{R}^n$ with the Euclidean inner product; an $n \times n$ matrix Q with complex entries is *unitary* if it satisfies the analogous condition. Prove that the following properties of a square matrix over $\mathbb{R}$ or $\mathbb{C}$ are equivalent:

(a) Q is unitary (if $F = \mathbb{C}$) or orthogonal (if $F = \mathbb{R}$).

(b) QQ^* is the identity matrix.

(c) The conjugate transpose Q^* is unitary (if $F = \mathbb{C}$) or orthogonal (if $F = \mathbb{R}$).

(d) The rows of Q form an orthonormal basis of F^n ($F = \mathbb{R}, \mathbb{C}$).

Here we are using the notation $Q^* = \overline{Q}^t$. (*Hint: you need to show that (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (a).*)