Worksheet 11/13. Math 110, Fall 2013. SOLUTIONS

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Gram-Schmidt; orthogonal projections

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.

1. a) Consider the Euclidean space $(\mathbb{R}^3, \cdot)$. Perform the Gram-Schmidt process on the following linearly independent list

$$(v_1, v_2, v_3) \stackrel{def}{=} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Find the point in $U = \text{span}(v_1, v_2)$ that is closest to the standard basis vector e_2 (distance is with respect to the dot product).

b) Consider the inner product space $(\mathbb{R}^3, \langle , \rangle)$, where

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2 + x_3y_3, \quad \text{for } \underline{x}, \underline{y} \in \mathbb{R}^3$$

- verify that this defines an inner product on $\mathbb{R}^3$ (*Hint: to show the 'positive definite' property ($\langle \underline{x}, \underline{x} \rangle \geq 0$) you will need to 'complete the square'.*)
- What is $||e_1||, ||e_2||, ||e_3||$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$, with respect to this inner product?
- Is the list (e_1, e_2, e_3) orthogonal with respect to this inner product?
- Find an orthonormal basis (z_1, z_2, z_3) of $\mathbb{R}^3$ (with respect to $\langle , \rangle$ above) such that $z_1 \in \text{span}(e_1)$, $z_1, z_2 \in \text{span}(e_1, e_2)$.
- Find the point in $W = \text{span}(e_1, e_2)$ that is closest to e_3 (distance is with respect to the norm induced by $\langle , \rangle$).

Solution: a) We will obtain an orthonormal basis (b_1, b_2, b_3):

$$b_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{2}} v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix},$$

$$b_2' = v_2 - v_2 \cdot b_1 b_1 = v_2 - \frac{1}{\sqrt{2}} b_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix} \implies b_2 = \frac{b_2'}{||b_2'||} = \frac{1}{\sqrt{3/2}} b_2' = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$$b_3' = v_3 - v_3 \cdot b_1 b_1 - v_3 \cdot b_2 b_2 = v_3 - \frac{1}{\sqrt{2}} b_1 + \frac{1}{\sqrt{6}} b_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} \implies b_3 = \frac{b_3'}{||b_3'||} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

b) This is the inner product defined by the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\langle \underline{x}, \underline{y} \rangle = \underline{x}^t A \underline{y}.$$

This implies the bilinearity properties of $\langle , \rangle$. Moreover, since A is symmetric (ie $A = A^t$) the we get $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$.

Suppose that $\langle \underline{x}, \underline{x} \rangle = 0$. Then, we have

$$0 = 2x_1^2 - 2x_1x_2 + x_2^2 + x_3^2 = 2\left(x_1 - \frac{x_2}{2}\right)^2 + \frac{x_2^2}{2} + x_3^2$$

so that

$$x_3 = 0,\ x_2 = 0,\ x_1 - \frac{x_2}{2} = 0 \implies x_1 = x_2 = x_3 = 0.$$

Moreover, we see that $\langle \underline{x}, \underline{x} \rangle \geq 0$, for every $\underline{x}$.

We have

$$||e_1|| = \sqrt{\langle e_1, e_1 \rangle} = \sqrt{2},\ ||e_2|| = \sqrt{\langle e_2, e_2 \rangle} = 1,\ ||e_3|| = \sqrt{\langle e_3, e_3 \rangle} = 1.$$

The list is not orthogonal since we have $\langle e_1, e_2 \rangle = -1 \neq 0$.

Applying Gram-Schmidt to the list (e_1, e_2, e_3) to obtain an orthonormal basis (z_1, z_2, z_3):

$$z_1 = \frac{e_1}{||e_1||} = \frac{1}{\sqrt{2}} e_1,$$

$$z_2' = e_2 - \langle e_2, z_1 \rangle z_1 = e_2 + \frac{e_1}{2} \implies z_2 = \frac{z_2'}{||z_2'||} = \sqrt{2}(e_2 + \frac{e_1}{2}),$$

$$z_3' = e_3 - \langle e_3, z_1 \rangle z_1 - \langle e_3, z_2 \rangle z_2 = e_3 \implies z_3 = \frac{z_3'}{||z_3'||} = e_3.$$

The point $w \in \text{span}(e_1, e_2)$ closest to e_3 is

$$w = \langle e_3, z_1 \rangle z_1 + \langle e_3, z_2 \rangle z_2 = 0.$$

2. Let $(v_1, v_2, v_3) \subset \mathbb{R}^3$ be linearly independent, where we are considering the Euclidean space $\mathbb{R}^3$ (ie, inner product space with inner product = dot product). Describe all orthonormal lists $(e_1, e_2, e_3) \subset \mathbb{R}^3$ such that $e_1 \in \text{span}(v_1)$. (*Hint: what are the possible choices for* e_1, e_2, e_3*?*)

Solution: Since we need $e_1 \in \text{span}(v_1)$ then we must have

$$e_1 = \pm \frac{v_1}{||v_1||}.$$

Now, we need to choose e_2, e_3 such that $e_2 \in \text{span}(v_1)^\perp$ and $e_3 \in \text{span}(v_1)^\perp$, and with $e_2 \cdot e_3 = 0$. Hence, we have that $\text{span}(v_1)^\perp$ is a two dimensional subspace of $\mathbb{R}^3$ (because $\dim \mathbb{R}^3 = \dim U + \dim U^\perp$, for any subspace U), ie, a plane, that is orthogonal to e_1. For any choice of unit length $e_2 \in \text{span}(v_1)^\perp$ there are precisely two vectors in this plane that are orthogonal to e_2 and have unit length. However, we need only that e_2 lies in the unit circle inside this plane $\text{span}(v_1)^\perp$. Thus, there are an infinite number of possible choices.

3. Consider the orthonormal list in Euclidean space $\mathbb{C}^3$

$$\left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -\sqrt{-1}/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ \sqrt{-1}/\sqrt{2} \end{bmatrix} \right)$$

Extend this list to an orthonormal basis of $\mathbb{C}^3$.

Find an orthonormal basis vector of null($\overline{A}^t$), where A is the 3×2 matrix with the above vectors as its columns. What do you notice? Can you explain this? (*Hint: adjoints!*)

Solution: There are a couple of ways to proceed - the easiest is to compute $\text{span}(v_1, v_2)^\perp$, where (v_1, v_2) are the orthonormal vectors listed above. Why? Since then we will have find the set of all vectors that are orthogonal to both v_1 and v_2. If we choose one of these, call it v_3 say, and such that $||v_3|| = 1$, then we have that (v_1, v_2, v_3) is an orthonormal list.

So, let's determine $\text{span}(v_1, v_2)$. We have that

$$\text{span}(v_1, v_2)^\perp = \{v \in \mathbb{C}^3 \mid v \cdot v_1 = 0,\ v \cdot v_2 = 0\}$$

and if $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{span}(v_1, v_2)^\perp$ then we must have

$$\frac{a}{\sqrt{2}} + \frac{c\sqrt{-1}}{\sqrt{2}} = 0,\ \frac{a}{\sqrt{2}} - \frac{c\sqrt{-1}}{\sqrt{2}} = 0.$$

Hence, we are looking for all solutions to

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{-1}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{-1}}{\sqrt{2}} \end{bmatrix} \underline{x} = \underline{0},$$

as the above equations are precisely captured in this previous matrix equation. Row-reducing the above matrix gives us the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that the solution set of the matrix equation is $\text{span}(e_2)$. Since e_2 has unit length we have an orthonormal basis

$$(v_1, v_2, e_2)$$

Notice that in our discussion we have determine $\text{null}(\overline{A}^t) = \text{span}(e_2)$. This holds because

$$\text{null}(T^*) = \text{range}(T)^\perp,$$

where $T \in L(\mathbb{C}^2, \mathbb{C}^3)$ is the operator defined by A, and $\text{range}(T) = \text{span}(v_1, v_2)$.

Adjoints; functionals

4. Consider the Euclidean space $\mathbb{C}^3$, and let $T \in L(\mathbb{C}^3)$ be defined by the matrix

$$A = \begin{bmatrix} \sqrt{-1} & -1 & 0 \\ 0 & \sqrt{-2}+1 & 1 \\ \sqrt{5} & -\sqrt{-1} & 0 \end{bmatrix}$$

(so that $T(\underline{x}) = A\underline{x}$). Determine the adjoint of T: that is, for any $\underline{w} \in \mathbb{C}^3$ what is $T^*(\underline{w})$?

Solution: We are considering the Euclidean inner product, and if $S = (e_1, e_2, e_3)$ is the standard orthonormal basis of $\mathbb{C}^3$, then we have that

$$[T]_S = A.$$

Hence, we have that

$$[T^*]_S = \overline{A}^t.$$

Therefore, we have that

$$T^*(w) = [T^*(w)]_S = [T]_S[w]_S = \overline{A}^t \underline{w}.$$

5. Let $(V, \langle , \rangle)$ be an inner product space, $T \in L(V)$. Suppose that $w \in \text{null}(T^*)$, $w \neq 0$. Show that $\text{range}(T) \subset \text{span}(w)^\perp$. By considering this result prove that T is an isomorphism if and only if T^* is an isomorphism. (Note: we are NOT saying that T^* is the inverse of T)

Solution: Let $v \in \text{range}(T)$. Then, $v = T(u)$, for some $u \in V$. Hence, we have

$$\langle v, w \rangle = \langle T(u), w \rangle = \langle u, T^*(w) \rangle = \langle u, 0 \rangle = 0 \in F.$$

Hence, we have that $v \in \text{null}(T^*)^\perp$ and since v is arbitrary, we have $\text{range}(T) \subset \text{span}(w)^\perp$. Suppose that T is an isomorphism. Then, we must have that $\text{range}(T) = V$. If $w \in \text{null}(T^*)$ then we have $V = \text{range}(T) \subset \text{span}(w)^\perp$, so that $\text{span}(w)^\perp = V$. In particular, $\langle w, w \rangle = 0$ so that $w = 0$. Hence, $\text{null}(T^*) = \{0\}$ and T^* is an isomorphism. To prove the other direction we replace T by T^*, and can show that is we assume T^* is an isomorphism then $(T^*)^* = T$ is an isomorphism.

6. Let $(V, \langle , \rangle_V)$ and $(W, \langle , \rangle_W)$ be inner product spaces, $T \in L(V, W)$. Suppose that $T^*T = I_V$ (the identity on V). Prove that $TT^* \in L(W)$ is the 'orthogonal projection onto $U = \text{range}(T)$' operator; that is, $TT^* = P_U$, where P_U is the orthogonal projection operator defined in Ch. 6 of Axler.

Solution: Suppose that $T^*T = I_V$. Let $w = T(v) \in \text{range}(T)$. Then, for any $z \in W$ we have

$$\langle TT^*(w), z \rangle = \langle TT^*T(v), z \rangle = \langle T(v), z \rangle \implies \langle TT^*(w) - T(v), z \rangle = 0.$$

Hence, we must have that $TT^*(w) = T(v) = w$, for any $w \in \text{range}(T) = U$. Hence, $\text{range}(TT^*) = \text{range}(T) = U$.

We also have that $\text{null}(T^*) = \text{range}(T)^\perp$. And we obtain

$$\text{null}(T^*) \subset \text{null}(TT^*) \subset \text{null}(T^*TT^*) = \text{null}(I_V T^*) = \text{null}(T^*).$$

Hence, $\text{range}(T)^\perp = \text{null}(T^*) = \text{null}(TT^*)$. Now, we need only show that $(TT^{ast})^2 = TT^*$ to show that $TT^* = P_U$, by some homework exercise. Indeed, we have

$$(TT^*)^2 = TT^*TT^* = T(I_V)T^* = TT^*.$$