# Worksheet 11/13. Math 110, Fall 2013.

*These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!*

**Gram-Schmidt; orthogonal projections**

*Throughout this worksheet $V$ will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.*

1. a) Consider the Euclidean space $(\mathbb{R}^3, \cdot)$. Perform the Gram-Schmidt process on the following linearly independent list

$$(v_1, v_2, v_3) \overset{def}{=} \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Find the point in $U = \text{span}(v_1, v_2)$ that is closest to the standard basis vector $e_2$ (distance is with respect to the dot product).

b) Consider the inner product space $(\mathbb{R}^3, \langle , \rangle)$, where

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + x_2y_2 + x_3y_3, \quad \text{for } \underline{x}, \underline{y} \in \mathbb{R}^3$$

- verify that this defines an inner product on $\mathbb{R}^3$ (*Hint: to show the 'positive definite' property ($\langle \underline{x}, \underline{x} \rangle \geq 0$) you will need to 'complete the square'.*)
- What is $||e_1||, ||e_2||, ||e_3||$, where $(e_1, e_2, e_3)$ is the standard basis of $\mathbb{R}^3$, with respect to this inner product?
- Is the list $(e_1, e_2, e_3)$ orthogonal with respect to this inner product?
- Find an orthonormal basis $(z_1, z_2, z_3)$ of $\mathbb{R}^3$ (with respect to $\langle , \rangle$ above) such that $z_1 \in \text{span}(e_1)$, $z_1, z_2 \in \text{span}(e_1, e_2)$.
- Find the point in $W = \text{span}(e_1, e_2)$ that is closest to $e_3$ (distance is with respect to the norm induced by $\langle , \rangle$).

2. Let $(v_1, v_2, v_3) \subset \mathbb{R}^3$ be linearly indpendent, where we are considering the Euclidean space $\mathbb{R}^3$ (ie, inner product space with inner product = dot product). Describe all orthonormal lists $(e_1, e_2, e_3) \subset \mathbb{R}^3$ such that $e_1 \in \text{span}(v_1)$. (*Hint: what are the possible choices for $e_1, e_2, e_3$?*)

3. Consider the orthonormal list in Euclidean space $\mathbb{C}^3$

$$\left( \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -\sqrt{-1}/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ \sqrt{-1}/\sqrt{2} \end{bmatrix} \right)$$

Extend this list to an orthonormal basis of $\mathbb{C}^3$.

Find an orthonormal basis vector of $\text{null}(\overline{A}^t)$. What do you notice? Can you explain this? (*Hint: adjoints!*)

**Adjoints; functionals**

4. Consider the Euclidean space $\mathbb{C}^3$, and let $T \in L(\mathbb{C}^3)$ be defined by the matrix

$$A = \begin{bmatrix} \sqrt{-1} & -1 & 0 \\ 0 & \sqrt{-2}+1 & 1 \\ \sqrt{5} & -\sqrt{-1} & 0 \end{bmatrix}$$

(so that $T(\underline{x}) = A\underline{x}$). Determine the adjoint of $T$: that is, for any $\underline{w} \in \mathbb{C}^3$ what is $T^*(\underline{w})$?

5. Let $(V, \langle , \rangle)$ be an inner product space, $T \in L(V)$. Suppose that $w \in \text{null}(T^*)$, $w \neq 0$. Show that $\text{range}(T) \subset \text{span}(w)^\perp$. By considering this result prove that $T$ is an isomorphism if and only if $T^*$ is an isomorphism. (Note: we are NOT saying that $T^*$ is the inverse of $T$)

6. Let $(V, \langle , \rangle_V)$ and $(W, \langle , \rangle_W)$ be inner product spaces, $T \in L(V, W)$. Suppose that $T^*T = I_V$ (the identity on $V$). Prove that $TT^* \in L(W)$ is the 'orthogonal projection onto $U = \text{range}(T)$' operator; that is, $TT^* = P_U$, where $P_U$ is the orthogonal projection operator defined in Ch. 6 of Axler.