Worksheet 10/09. Math 110, Fall 2013.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Invariant Subspaces:

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.

1. Let $V = \mathbb{R}^2$ (over $F = \mathbb{R}$), $T \in L(V)$. Suppose that T admits an eigenvector v with eigenvalue $\lambda \in \mathbb{R}$. Let $u \notin \text{span}(v)$; explain why (u, v) is a basis of $\mathbb{R}^2$. Suppose that $T(u) = au + bv$, with $b \neq 0$. Prove that a is also an eigenvalue of T.

Solution: (u, v) is a basis since it is a linearly independent list in $\mathbb{R}^2$ - if it were linearly dependent then we would have $u \in \text{span}(v)$ or one of u, v would be equal to 0. With respect to the basis $B = (v, u)$ of $\mathbb{R}^2$ we have

$$[T]_B = \begin{bmatrix} \lambda & b \\ 0 & a \end{bmatrix},$$

so that a is also an eigenvalue of T - eigenvalues of T always appear on the diagonal of any upper-triangular matrix representation of T.

2. In Q1 suppose that $\lambda = a \neq 0$ so that T has only one distinct eigenvalue. Determine $w \in \mathbb{R}^2$ such that $T(w) = \lambda w + v$. In this case, what are the only T-invariant subspaces of $\mathbb{R}^2$? (Don't forget the obvious ones!)

Solution: The assumption implies that

$$[T]_B = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix},$$

with $\lambda \neq 0$, $b \neq 0$. We are trying to solve $(T - \lambda)(w) = v$, for w. In terms of matrices, we are trying to solve

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We find the solution $\underline{w} = \begin{bmatrix} 0 \\ 1/b \end{bmatrix}$. Hence, we take $w = \frac{1}{b} u$.

The only T-invariant subspaces are $\{0\}, \mathbb{R}^2$ and $\text{span}(v)$.

3. Let (v, w) be as in Q2. Prove that (v, w) is a basis of $\mathbb{R}^2$ (ie, $w \notin \text{span}(v)$). What is the matrix of T with respect to (v, w)

Solution: The matrix of T is

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

4. Let $T \in L(V)$ and $U, W \subset V$ be T-invariant subspaces. Prove that $U + W$ is T-invariant.

Solution: Let $u + w \in U + W$, so that $u \in U$ and $w \in W$. Then, $T(u) \in U$ and $T(w) \in W$. Hence, $T(u + w) = T(u) + T(w) \in U + W$.

5. Let $T \in L(\mathbb{R}^2)$. Suppose that T admits two distinct proper T-invariant subspaces of $\mathbb{R}^2$ ($U \subset V$ is a *proper subspace of V* if U is a subspace and $U \neq \{0\}, V$). Prove that there is a basis of $\mathbb{R}^2$ such that the matrix of T with respect to this basis is diagonal.

Solution: Since T admits two distinct proper T-invariant subspaces, let's call them L_1, L_2, they must be T-invariant lines in $\mathbb{R}^2$. Hence, if $L_1 = \text{span}(u_1)$, $L_2 = \text{span}(u_2)$, then $T(u_1) = c_1 u_1$, for some c_1, and $T(u_2) = c_2 u_2$, for some c_2. Hence, using that $L_1 \neq L_2$, we must have $L_1 \cap L_2 = \{0\}$, so that $\mathbb{R}^2 = L_1 \oplus L_2$ (use dimension formula on $L_1 + L_2$ to deduce that $L_1 + L_2 = \mathbb{R}^2$. Hence, we can take the basis to be (u_1, u_2) so that the matrix is

$$\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

6. Let $T \in L(\mathbb{R}^3)$ be such that there is some $0 \neq v \in \mathbb{R}^3$ with $T^3(v) = \underline{0}$, while $T^2(v) \neq \underline{0}$.

- show that $T(v) \neq \underline{0}$.
- show that the only eigenvalue of T is $\lambda = 0$.
- show that $B = (v, T(v), T^2(v))$ is linearly independent, hence a basis of $\mathbb{R}^3$.
- what is the matrix of T with respect to B?

Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \; ; \; \underline{v} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \underline{v}$$

Prove that there is a basis of $\mathbb{R}^3$ such that the matrix of T with respect this basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: if $T(v) = 0$, then we would have $T^2(v) = 0$. Hence, $T(v) \neq 0$. Since $T^2(v) \neq 0$ and $T(T^2(v)) = 0$, we know that T admits an eigenvalue 0. If λ is any eigenvalue of T then we must have some corresponding eigenvector u, so that $T(u) = \lambda u$. Now, we always have

$$\text{null}(T) \subset \text{null}(T^2) \subset \text{null}(T^3),$$

and we have that $v \in \text{null}(T^3)$ but $v \notin \text{null}(T^2)$. Also, $T(v) \in \text{null}(T^2)$, while $T(v) \notin \text{null}(T)$ (else, $T^2(v) = 0$). Hence, we have shown that the inclusions above are 'strict', so that $\text{null}(T) \neq \text{null}(T^2)$ and $\text{null}(T^2) \neq \text{null}(T^3)$. Hence, since $\dim \text{null}(T) \geq 1$ (we have a nonzero element $T^2(v) \in \text{null}(T)$), then

$$1 \leq \dim \text{null}(T) < \dim \text{null}(T^2) < \dim \text{null}(T^3) \leq 3 = \dim \mathbb{R}^3.$$

The only way this can occur is if

$$\dim \text{null}(T) = 1, \; \dim \text{null}(T^2) = 2, \; \dim \text{null}(T^3) = 3.$$

In particular, $\text{null}(T^3) = \mathbb{R}^3$. Hence, we find that

$$0 = T^3(u) = \lambda^3 u \implies \lambda = 0.$$

We have that $T^2(v) \in \text{null}(T)$, while $T(v) \in \text{null}(T^2)$ and $T(v) \notin \text{null}(T)$. Hence, the list $(T(v), T^2(v))$ is linearly independent. Furthermore, $v \in \text{null}(T^3)$, but $v \notin \text{null}(T^2)$.Since $(T(v), T^2(v))$ is a linearly independent list in $\text{null}(T^2)$, this implies that $(v, T(v), T^2(v))$ is linearly independent. Hence, it is a basis of $\mathbb{R}^3$. Finally the matrix is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$