Worksheet 9/18. Math 110, Fall 2013. Solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Linear Maps:

Let $T : V \to W$ be a linear map. Recall that we have the subspaces

$$\operatorname{null} T = \{v \in V \mid T(v) = 0_W\} \subset V,$$

$$\operatorname{range}(T) = \{w \in W \mid w = T(v), \text{ for some } v \in V\} \subset W.$$

1. Let V be finite dimensional vector space, W arbitrary vector space. Show that if $(v_1, \dots, v_n)$ is a basis of V and $w_1, \dots, w_n \in W$ are arbitrary, then there is a linear map $T : V \to W$ such that $T(v_i) = w_i$, for each i, ie, define $T(v) \in W$ for any input $v \in V$ (there's not much (=zero) choice here!). Suppose that $S : V \to W$ is a linear map such that $S(v_i) = w_i$, for each i. Show that $T = S$ (what does it mean for two functions to be equal?).

Hence, you've shown that a linear map is uniquely determine by what it does to a basis of V (this is dependent on the basis you have chosen!).

Solution: Since we want T to be linear we must necessarily have

$$T(\sum_i a_i v_i) = \sum_i a_i T(v_i) = \sum_i a_i w_i, \; a_i \in F$$

Hence, as $(v_1, \dots, v_n)$ is a basis of V every $v \in V$ can be written in a unique way as $v = \sum_i a_i v_i$, for some $a_i \in F$. Hence, we define

$$T(v) = \sum_i a_i w_i.$$

This, is a linear map (check!) by construction.

Now, if $S : V \to W$ is any other linear map such that $S(v_i) = w_i$ then, for $v = \sum_i a_i v_i$, we have

$$S(v) = \sum_i a_i S(v_i) = \sum_i a_i w_i = \sum_i a_i T(v_i) = T(v),$$

since both S and T are linear. Thus, $S(v) = T(v)$, for every $v \in V$, so that they must be equal.

2. Suppose that you have a linear map $T : \mathbb{R}^5 \to \mathbb{R}^{10}$. What are the possible values for $\dim \operatorname{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \operatorname{null}(T)$? Define a linear map that realises each of these possible values (Hint: you've already done the work!)

Solution: We use the dimension formula: if $T : V \to W$ is a linear map, with V finite dimensional, then

$$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{range}(T).$$

Hence, $\dim \operatorname{range}(T) \leq \dim \mathbb{R}^5 = 5$, and the possible values of $\dim \operatorname{range}(T)$ are $0, 1, 2, 3, 4, 5$. Using Q1 we can define a linear map by specifying the images of a basis $(v_1, \dots, v_n)$ of $\mathbb{R}^5$.

Take the standard basis $(e_1, \dots, e_5)$. Then, define $T_i \in L(\mathbb{R}^5, \mathbb{R}^{10})$, for $i = 0, \dots, 5$, to be a linear map such that $\dim \operatorname{range}(T_i) = i$, as follows:

$$T_i(e_j) = \begin{cases} 0 \in \mathbb{R}^{10}, & \text{if } j > i, \\ e_j \in \mathbb{R}^{10}, & \text{if } j \leq i. \end{cases}$$

The possible values of $\dim \operatorname{null}(T)$ are $0, 1, 2, 3, 4, 5$, as $5 = \dim \mathbb{R}^5 \geq \dim \operatorname{null}(T)$. Moreover, we see that $\dim \operatorname{null}(T_i) = 5 - i$.

3. Suppose that you have a linear map from $T : \mathbb{R}^7 \to \mathbb{R}^4$. What are the possible values of $\dim \operatorname{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \operatorname{null}(T)$? Define a linear map that realises each of these possible values.

Solution: This question is similar to the previous one, except we must now have $\operatorname{range}(T) \subset \mathbb{R}^4$ and $\dim \operatorname{range}(T) \leq \dim \mathbb{R}^7 = 7$ so that the allowed possibilities for $\dim \operatorname{range}(T)$ are $0, 1, 2, 3, 4$. We define T_i, for $i = 0, 1, 2, 3, 4$, in the same way as the previous question. Then, $\dim \operatorname{range}(T_i) = i$. The possible values of $\dim \operatorname{null}(T)$ are now $3, 4, 5, 6, 7$, since we require that $\dim \operatorname{null}(T) = \dim \mathbb{R}^7 - \dim \operatorname{range}(T) = 7 - \dim \operatorname{range}(T)$. The map T_i then has $\dim \operatorname{null}(T_i) = 7 - i$.

4. (Harder) Let U, V, W be vector spaces and $S \in L(V, W)$. Consider the function

$$f_S : L(U, V) \to L(U, W) \; ; \; T \mapsto S \circ T.$$

Show that f_S is a linear map.

Suppose now that $U = V = \mathbb{R}^2$, $W = \mathbb{R}^3$ and

$$S : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}$$

Show that S and f_S are injective.

What about if we take

$$S : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \underline{x};$$

is S or f_S injective?

Prove, for arbitrary U, V, W and $S \in L(V, W)$: S is injective if and only if f_S is injective.

Solution: Let $T, T' \in L(U, V)$ and $\lambda, \mu \in F$. Then, $f_S(\lambda T + \mu T') \in L(U, W)$ is the linear map with domain U and codomain W, and

$$f_S(\lambda T + \mu T')(u) = (\lambda T + \mu T') \circ S(u) = (\lambda T(S(u))) + \left(\mu T'(S(u))\right)$$

$$\lambda (T \circ S)(u) + \mu (T' \circ S)(u) = \left(\lambda f_S(T) + \mu f_S(T')\right)(u) \implies f_S(\lambda T + \mu T') = \lambda f_S(T) + \mu f_S(T')$$

and f_S is linear.

Let A be the first matrix appearing above. Then, since the columns of A are linearly independent (they are not multiples of each other) we have that S is injective - else there is some nonzero $\underline{x} \in \mathbb{R}^2$ such that

$$\underline{0} = A\underline{x} = x_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

which contradicts the linear independence of the columns. Now, any linear map from $U = \mathbb{R}^2$ to $V = \mathbb{R}^2$ is of the form

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \ ; \ \underline{x} \mapsto B\underline{x},$$

where $B = [\underline{b}_1 \ \underline{b}_2]$ is the 2×2 matrix with real entries and $\underline{b}_i = T(\underline{e}_i)$. Suppose that $f_S(T) = 0 \in L(U, W)$ is the zero transformation. Thus, in particular, for the standard basis vectors $\underline{e}_1, \underline{e}_2$ of $\mathbb{R}^2 = U$, we have

$$\underline{0} = f_S(T)(\underline{e}_1) = S(T(\underline{e}_1)) = S(B\underline{e}_1) = A\underline{b}_1 \implies \underline{b}_1 = \underline{0},$$

since S is injective. Similarly, we find that $\underline{b}_2 = \underline{0}$ so that $T = 0 \in L(U, V)$ is the zero transformation.

For the second matrix, we have that S is not injective - you can check that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \text{null}(S)$, so that $\text{null}(S) \neq \{0\}$. Also, f_S is not injective since if we let

$$B \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then $T : \mathbb{R}^2 \to \mathbb{R}^2 \ ; \ \underline{x} \mapsto B\underline{x}$ satisfies $ST = 0 \in L(U, W)$ - indeed, we have for any $\underline{x} \in \mathbb{R}^2 = U$,

$$S(T(\underline{x})) = AB\underline{x} = \underline{0},$$

so that ST is the zero linear map.

($\Rightarrow$) Suppose that S is injective. Then, we have that $\text{null}(S) = \{0\}$. Suppose that $T \in \text{null}(f_S)$. Thus, $ST = 0 \in L(U, W)$ is the zero transformation. Let $u \in U$. Then,

$$0 = ST(u) = S(T(u)) \implies T(u) \in \text{null}(S) = \{0\} \implies T(u) = 0, \quad \text{for every } u \in U.$$

Hence, we must have that T is the zero transformation.

($\Leftarrow$) Suppose that f_S is injective. Then, we have $\text{null}(f_S) = \{0\} \subset L(U, V)$. Let $v \in \text{null}(S) \subset V$. We want to show that $v = 0 \in V$. If $U = \{0\}$ then there is nothing to show (as then $L(U, V) = \{0\}$ consists of only the zero transformation). So, assume that $\dim U \geq 1$. Let $u \in U$ be nonzero and extend to a basis $(u, u_2, \ldots, u_k)$ of U. Define $T \in L(U, V)$ as follows

$$T(u) = v, T(u_i) = 0.$$

Then, for any $z \in U$ we have $z = au + \sum_{i \geq 2} a_i u_i$ and

$$S(T(z)) = S(aT(u)) = S(av) = aS(v) = 0 \in W.$$

Hence, ST is the zero transformation, ie, $f_S(T) = 0$. Hence, $T = 0$ is the zero transformation so that $0 = T(u) = v$. Thus, $\text{null}(S) = \{0\}$ and S is injective.