Worksheet 9/18. Math 110, Fall 2013.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Linear Maps:

Let $T : V \to W$ be a linear map. Recall that we have the subspaces

$$\text{null}\, T = \{v \in V \mid T(v) = 0_W\} \subset V,$$

$$\text{range}(T) = \{w \in W \mid w = T(v), \text{ for some } v \in V\} \subset W.$$

1. Let V be finite dimensional vector space, W arbitrary vector space. Show that if $(v_1, \dots, v_n)$ is a basis of V and $w_1, \dots, w_n \in W$ are arbitrary, then there is a linear map $T : V \to W$ such that $T(v_i) = w_i$, for each i, ie, define $T(v) \in W$ for any input $v \in V$ (there's not much (=zero) choice here!). Suppose that $S : V \to W$ is a linear map such that $S(v_i) = w_i$, for each i. Show that $T = S$ (what does it mean for two functions to be equal?).

Hence, you've shown that a linear map is uniquely determine by what it does to a basis of V (this is dependent on the basis you have chosen!).

2. Suppose that you have a linear map $T : \mathbb{R}^5 \to \mathbb{R}^{10}$. What are the possible values for $\dim \text{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \text{null}(T)$? Define a linear map that realises each of these possible values (Hint: you've already done the work!)

3. Suppose that you have a linear map from $T : \mathbb{R}^7 \to \mathbb{R}^4$. What are the possible values of $\dim \text{range}(T)$? Define a linear map that realises each of these possible values. What are the possible values of $\dim \text{null}(T)$? Define a linear map that realises each of these possible values.

4. (Harder) Let U, V, W be vector spaces and $S \in L(V, W)$. Consider the function

$$f_S : L(U, V) \to L(U, W)\ ;\ T \mapsto S \circ T.$$

Show that f_S is a linear map.

Suppose now that $U = V = \mathbb{R}^2$, $W = \mathbb{R}^3$ and

$$S : V \to W\ ;\ \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}$$

Show that S and f_S are injective.

What about if we take

$$S : V \to W\ ;\ \underline{x} \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \underline{x};$$

is S or f_S injective?

Prove, for arbitrary U, V, W and $S \in L(V, W)$: S is injective if and only if f_S is injective.