Worksheet 9/11. Math 110, Fall 2013. Solutions.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Subspaces, Sums, Bases:

1. Verify that the following sets U are a vector subspace of the given vector space V:

i) Consider the $\mathbb{R}$-vector space $V = Mat_3(\mathbb{R}) = \{3 \times 3 \text{ matrices with real entries}\}$
$U_1 = \{A \in V \mid A^t = A\}$ *Note: if* $A = [a_{ij}]$ *then* $A^t = [a_{ji}]$ *is the transpose of* A*;*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \implies A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

The matrices in U_1 are called symmetric.

ii) $V = Mat_3(\mathbb{R})$
$U_2 = \{A \in V \mid A^t = -A\}$.

The matrices in U_2 are called antisymmetric.

Solution: i) Let $A, B \in U_1$ and $\lambda, \mu \in F$. Then, we have that

$$\lambda A + \mu B = [\lambda a_{ij} + \mu b_{ij}]$$

ie, the ij-entry of the matrix $\lambda A + \mu B$ is $\lambda a_{ij} + \mu b_{ij}$. Thus, we have

$$(\lambda A + \mu B)^t = [\lambda a_{ji} + \mu b_{ji}] = \lambda [a_{ji}] + \mu [b_{ji}] = \lambda A^t + \mu B^t = \lambda A + \mu B,$$

since $A, B \in U_1$. Hence, we have shown that U_1 is a subspace.

ii) This is a similar calculation as the one above.

2. Show that $U_1 \cap U_2 = \{0\}$ and that $\dim U_1 = 6$. Give a basis of U_2. What is $\dim U_2$? Conclude that $Mat_3(\mathbb{R}) = U_1 \oplus U_2$.

Does anything change if we consider the $\mathbb{C}$*-vector space* $V = Mat_3(\mathbb{C})$*? What about if we consider the* $\mathbb{R}$*-vector space* $Mat_3(\mathbb{C})$*? (Hint: For the latter case the only thing that changes is the dimension and basis.)*

Solution: Let $A \in U_1 \cap U_2$. Then, we have that $A^t = A$ and $A^t = -A$. Hence, we have $A = -A$ so that, for each (i, j), $a_{ij} = -a_{ij} \implies a_{ij} = 0$, since each $a_{ij} \in \mathbb{R}$. Therefore, $A = 0$ and $U_1 \cap U_2 = \{0\}$.

If $A \in U_1$ then we must have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = A^t$$

so that

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = aE_{11} + b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + dE_{22} + e(E_{23} + E_{32}) + fE_{33}$$

where E_{ij} is the 3×3 matrix with a 1 in the ij-entry and 0s elsewhere. Hence,

$$(E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32})$$

is a spanning list of U_1. Moreover, if we have a linear relation

$$aE_{11} + b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + dE_{22} + e(E_{23} + E_{32}) + fE_{33} = 0 \in U_1$$

then we have

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = 0 \in U_1,$$

so that $a = b = c = d = e = f = 0$. Hence, the list given above is linearly indendent and a spanning list so is a basis of U_1. Moreover, we see that $\dim U_1 = 6$. In a similar way we have that

$$(E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32})$$

is a spanning list of U_2 and linearly independent. Hence, it is a basis of U_2 and $\dim U_2 = 3$. Now, $U_1 + U_2 \subset Mat_3(\mathbb{R})$ is a subspace, and we can use the dimension formula to conclude that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim U_1 \cap U_2 = 6 + 3 - 0 = 9 = \dim Mat_3(\mathbb{R})$$

Hence, $U_1 + U_2 = Mat_3(\mathbb{R})$. Moreover, the sum is direct since $U_1 \cap U_2 = \{0\}$. Hence, $U_1 \oplus U_2 = Mat_3(\mathbb{R})$.

3. *(Harder)* Let J be the matrix

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Consider the subset

$$U = \{A \in Mat_4(\mathbb{C}) \mid A^t J + JA = 0\}$$

Show that U is a subspace of the $\mathbb{C}$-vector space $V = Mat_4(\mathbb{C})$ and that $\dim U = 10$. Determine an explicit basis of U. *Hint: for the last part, write down an arbitrary 4×4 matrix A and determine what the condition $A^t J + JA = 0$ implies for the entries of A. For example, if $A = [a_{ij}$ (i's are rows and j's are columns!), then you should find that $a_{11} = -a_{44}$ and $a_{13} = a_{24}$.*

Consider the subspace

$$W = \left\{ \begin{bmatrix} a & b & c & 0 \\ d & e & 0 & -c \\ f & 0 & e & b \\ 0 & -f & d & a \end{bmatrix} \mid a, b, c, d, e, f \in \mathbb{C} \right\}$$

Show that $Mat_4(\mathbb{C}) = U \oplus W$. *(Hint: you need to show that $U \cap W = \{0\}$ and that $\dim U \oplus W = Mat_4(\mathbb{C})$. Think about why this suffices to give the claim...)*

Solution: Let $A, B \in U$ and $\lambda, \mu \in \mathbb{C}$. Then, we have

$$(\lambda A + \mu B)^t J + J(\lambda A + \mu B) = \lambda(A^t J + JA) + \mu(B^t J + JB) = 0 + 0 = 0$$

where we have used that $A, B \in U$ so that $A^t J + JA = 0, B^t J + JB = 0$. Hence, U is a subspace.

Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \in U$$

Thus, we have

$$A^t J = -JA \implies \begin{bmatrix} -a_{41} & -a_{31} & a_{21} & a_{11} \\ -a_{42} & -a_{32} & a_{22} & a_{12} \\ -a_{43} & -a_{33} & a_{23} & a_{13} \\ -a_{44} & -a_{34} & a_{24} & a_{14} \end{bmatrix} = \begin{bmatrix} -a_{41} & -a_{42} & -a_{43} & -a_{41} \\ -a_{31} & -a_{32} & -a_{33} & -a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

Hence, we must have, for example,

$$a_{31} = a_{42}, \ a_{22} = -a_{33}, \ a_{13} = a_{24}, \ \text{etc.}$$

Hence,

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & c \\ h & i & -f & -b \\ j & h & -e & -a \end{bmatrix}$$

so that

$$(E_{11} - E_{44}, E_{22} - E_{33}, E_{13} + E_{24}, E_{23}, E_{14}, E_{12} - E_{34}, E_{21} - E_{43}, E_{31} + E_{42}, E_{41}, E_{32})$$

is a basis of U. Hence, $\dim U = 10$.

It is straightforward to see that $\dim W = 6$ (ie, six degrees of freedom in defining W). If $A = [a_{ij}] \in U \cap W$ then A must be a matrix of the type described in W and U (as we obtained above). In particular we should have that the $a_{11} = -a_{44}$ (since $A \in U$) and $a_{11} = a_{44}$ - hence, we'd require that $a_{44} = -a_{44} \implies a_{44} = 0 = a_{11}$. In this way we see that the only possible such matrix is $A = 0$. Hence, $U \cap W = \{0\}$. Now, as $\dim(U + W) = 10 + 6 = 16 = \dim Mat_4(\mathbb{C})$, using the dimension formula, we must have $U + W = Mat_4(\mathbb{C})$. Since $U \cap W = \{0\}$ this sum is a direct sum.