Worksheet 9/11. Math 110, Fall 2013.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Subspaces, Sums, Bases:

1. Verify that the following sets U are a vector subspace of the given vector space V:

   i) Consider the $\mathbb{R}$-vector space $V = Mat_3(\mathbb{R}) = \{3 \times 3 \text{ matrices with real entries}\}$
   $U_1 = \{A \in V \mid A^t = A\}$ *Note: if $A = [a_{ij}]$ then $A^t = [a_{ji}]$ is the transpose of A;*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \implies A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

   The matrices in U_1 are called symmetric.

   ii) $V = Mat_3(\mathbb{R})$
   $U_2 = \{A \in V \mid A^t = -A\}$.

   The matrices in U_2 are called antisymmetric.

2. Show that $U_1 \cap U_2 = \{0\}$ and that $\dim U_1 = 6$. Give a basis of U_2. What is $\dim U_2$? Conclude that $Mat_3(\mathbb{R}) = U_1 \oplus U_2$.

Does anything change if we consider the $\mathbb{C}$-vector space $V = Mat_3(\mathbb{C})$? What about if we consider the $\mathbb{R}$-vector space $Mat_3(\mathbb{C})$? (Hint: For the latter case the only thing that changes is the dimension and basis.)

3. *(Harder)* Let J be the matrix

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Consider the subset

$$U = \{A \in Mat_4(\mathbb{C}) \mid A^t J + JA = 0\}$$

Show that U is a subspace of the $\mathbb{C}$-vector space $V = Mat_4(\mathbb{C})$ and that $\dim U = 10$. Determine an explicit basis of U. *Hint: for the last part, write down an arbitrary 4×4 matrix A and determine what the condition $A^t J + JA = 0$ implies for the entries of A. For example, if $A = [a_{ij}$ (i's are rows and j's are columns!), then you should find that $a_{11} = -a_{44}$ and $a_{13} = a_{24}$.*

Consider the subspace

$$W = \left\{ \begin{bmatrix} a & b & c & 0 \\ d & e & 0 & -c \\ f & 0 & e & b \\ 0 & -f & d & a \end{bmatrix} \mid a, b, c, d, e, f \in \mathbb{C} \right\}$$

Show that $Mat_4(\mathbb{C}) = U \oplus W$. *(Hint: you need to show that $U \cap W = \{0\}$ and that* $\dim U \oplus W = Mat_4(\mathbb{C})$. *Think about why this suffices to give the claim...)*