Koszul duality: Qiaochu Yuan

- Koszul algebras and Koszul duality.

- $k$ - field
- $V/k$ - f.d. vector space
- $R \in V \otimes V$

$A = T(V) / \langle R \rangle$ "quadratic ring"

- $R = \text{Sym}^2(V) \Rightarrow A = \Lambda(V)$
- $R = A_{1+2}(V) \Rightarrow A = S(V)$

Denote $R^* = \{ f \in V^* \otimes V^* \mid f(r) = 0, \forall r \in R \} \subseteq V^* \otimes V^*$

Define $A^0 = T(V^*) / \langle R^* \rangle$ "quadratic dual" of $A$

$S(V)^! \cong \Lambda(V^*)$ and $(-)^!$ involutive in this case.

Would like to understand cohomology for various functors out of $A$-mod etc.

Koszul (resolution): $A = \bigoplus_{i>0} A_i$, $A_0 = k$.

$A \rightarrow k$, want to resolve $A^k$ by free $A$-mods

$A \otimes R \rightarrow A \otimes V \rightarrow A \rightarrow k$

$A \otimes (R \otimes V \otimes R)$

$A \otimes (V^2 \otimes R \otimes V \otimes R \otimes V^2)$
\[ \text{Def:} \quad A \text{ is Koszul if the Koszul complex is a resolution of } k. \]

\[ S(V) \text{ is Koszul:} \]

\[ \cdots \to S(V) \otimes V^3 \to S(V) \otimes V^2 \to S(V) \to k \]

(\text{"classical" Koszul complex})

\[ \bigcap_{i=1}^n (A_i^*)^* \supseteq (A_i^*)^* \]

(\text{shifted by } R^*)

\[ 
\]

\[ \text{Koszul complex:} \]

\[ \cdots \to A \otimes (A_i^*)^* \to A \otimes (A_i^*)^* \to A \otimes (A_0^*)^* \to k \]

(\text{K.})

\[ \text{obtain a double complex.} \]

\[ \]

\[ \cdots \to A_0 \otimes (A_0^*)^* \to A_0 \otimes (A_0^*)^* \to \bullet \]

\[ \]

\[ \text{If } K_0 \text{ is a resolution then isomorphism is concentrated in bottom right; computing the cohomology of this bicomplex in } 2 \text{ ways gives} \]

\[ \text{Prop:} \quad \text{If } A \text{ Koszul then } A^! \text{ Koszul. "Koszul dual"} \]
Monte theory for mgs:

Let $R, S$ be mgs.

Question: When is $R\text{-mod} \cong S\text{-mod}$?

\[ \text{distinguished} \]
\[ R \]
\[ (M \rightarrow \text{Hom}_R(R, M)) \text{ is} \]
\[ \text{faithful functor} \]
\[ R \rightarrow R \rightarrow M \rightarrow 0 \]
for such $R$, $R$ is a
\[ \text{"generator"} \]
\[ \text{Hom}_R(R, -) \text{ preserves (finite) coproducts} \]
\[ \text{exacts (} R \text{ small/compact)} \]

Theorem:

Let $A$ be an abelian category, $P$ small projective generator.

Then, $\text{Hom}(P, -) : A \rightarrow \text{mod-End}_A(P, P)$.

is an equivalence of categories.

Corollary: $R\text{-mod} \cong S\text{-mod}$

\[ \iff R\text{-mod contains a small proj. generator} \]
\[ P \text{ with } \text{End}(P) \cong S^{op} \]
Key: \( R^* \in R\text{-mod} \)
\[ \text{End}(R) \cong \text{Mat}_n(R^{op}) \]
\[ \Rightarrow \ R\text{-mod} \cong \text{Mat}_n(R)\text{-mod}. \]

- Derived Morita Theory:

\[ \text{Q}^\ast: \text{When are } D^2(R) \cong D^2(S)? \] (as triangulated categories)

Recall: \( A \) - abelian category

\[ \text{Ch}(A) - \text{chain complexes in } A. \]

\[ \Downarrow \]

\[ K(A) - \text{homotopy category} \]

\[ \Downarrow \]

\[ D(A) - "\text{formally invert qis}" \]

Theorem: \( R, S \) are derived Morita equivalent

\[ \iff \ D(R) \text{ contains a tilting complex } T \]

with

\[ R \text{End}(T) \cong S^{op}, \]

and equivalence given by \( R\text{Hom}(T, -) \).

- tilting complex \( \iff \) qis to bounded ex of f.g. projective modules ("small")

- coprods, cokernels, shifts, \( \Delta \)'s

("generator")
Theorem: \( A \) graded ungl, \( A = \bigoplus_{i \geq 0} A_i \). \( \dim A < \infty \)

Then, \( \mathbf{RHom} (k, -) \) gives a derived Morita equivalence from

\[ D^b (A \text{-gmod}) \to D^b (A^! \text{-gmod}) \]

\[ \uparrow \quad \text{f.g.} \]

We call \( K = \mathbf{RHom} (k, -) \) the \textbf{Koszul duality functor}.

\[ \to \text{BGG correspondence. (next week)} \]