- Bucher, Gunning, Soergel: "Koszul duality patterns..."
- Berenstein, Gelfand, Gelfand: "Algebraic vector bundles..."
- Macaulay 2

How Tate resolutions compute cohomology of coherent sheaves on projective spaces.

1. Setup of Koszul duality:

\[ V = k[x_1, ..., x_n] \quad \text{and} \quad V^* = k[e_1, ..., e_m], \quad \langle e_i, x_j \rangle = \delta_{ij} \]

\[ R \leq V \otimes_k V \implies A = T(V) / (R) \]

\[ R^* \leq V^* \otimes_k V^* \implies R^* \]

\[ A^! = T(V^*) / (R^*) \]

\[ A^! = \bigoplus_{d \geq 0} A^!_d \quad \text{deg} = -d. \]

Consider the 'graded dual' of \( A' \):

\[ (A')^\circ = \bigoplus_{d \geq 0} (A^!)^*_d \]

Let \( N \) be graded \( A^! \)-module, and consider (left)
\[ a \otimes n \longrightarrow \sum_i a_i \otimes e_i \otimes n \]

\[ F(N): \cdots \longrightarrow A(i) \otimes N_i \longrightarrow A(i+i) \otimes N_{i+i} \longrightarrow \cdots \]

Consider \( N_i \) as (ungraded) \( \mathbb{V} \) s.

Set grading comes from \( \mathbb{LH} \) term.

Suppose \( N \) is a complex

\[ \cdots \longrightarrow N^j \longrightarrow N^{j+1} \longrightarrow \cdots \]

and consider the double complex

\[ \cdots \longrightarrow A(i) \otimes N^j_i \longrightarrow A(i+i) \otimes N^j_{i-1} \longrightarrow \]

\[ \cdots \longrightarrow A(i) \otimes N_{i+i} \longrightarrow A(i+i) \otimes N_{i-1} \longrightarrow \]

(direct sum)

\[ \Rightarrow F(N) = \text{total complex of double ex} \]

For \( M \) a graded \((\text{left})\) \( A \)-module,

\[ G(M): \cdots \longrightarrow \text{Hom}_k(A^i(i), M_i) \longrightarrow \text{Hom}_k(A^{i+i}(i+i), M_{i+i}) \longrightarrow \cdots \]

graded hom.

\[ (A^{\hat{i}} \otimes (-i)) \otimes_k M_i \]

For \( M \) ex \( \Rightarrow G(M) \) total (direct product) ex.

of double ex.
\[
\begin{align*}
F(N) &= \text{Sym}_A(V) = S(A) \\
G(M) &= \text{Hom}_A(A \otimes A, M) \\
\end{align*}
\]

"adjunction is tensor-hom adjunction"

**Tate Resolutions**

- Now, \( A = \text{Sym}_k(V) = S(V) \)
  \[
  A^i = E(V^i) \\
  (A^i)^\otimes = \bigoplus_{i=0}^{\dim V} \Lambda^i V
  \]

Note: \( \stackrel{\wedge}{E} \sim E \) as left \( E \)-module; \( \wedge \) "self-injective."

- For \( N \) graded \( \text{fg} \) \( (\text{left}) \) \( E \)-module.

Free resolution: \( N \to T^2 \to T^1 \to T^0 \to N \quad (1) \)

Cofree (injective) resolution: \( N \to T' \to T^2 \to \cdots \quad (2) \)

where \( T^i = \bigoplus_{j \in \mathbb{Z}} \stackrel{\wedge}{E}(-j) \) \( \delta^i \).

"Splice together: \( N \to T^2 \to T^1 \to T^0 \to T^1 \to T^2 \to \cdots \)

"TATE RESOLUTION."

Properties:
1) Doubly infinite \( (\mathcal{N} \text{ not project}) \)
2) Unique up to htpy
3) Exact everywhere

Assume (1), (2) are minimal free/cofree resolutions (i.e. \( T \& T^* \) have trivial differentials)

4) Unique up to isomorphism
5) The whole \( \mathcal{C} \) is determined by any of its differentials

Tate Resolution:

Everywhere exact \( \mathcal{C} \) of graded free \( E \)-modules.

\( \text{Tate}(E) = \text{htpy category} \)

\( \text{Tate}^{\text{min}}(E) = \text{htpy category of min. Tate res} \)

3. Coherent sheaves

Let \( \mathcal{F} \) be coherent sheaf on \( \mathcal{P}(V) = \text{Proj}(S(V)) \)

\( M = \bigoplus_{d \in \mathbb{Z}} \mathcal{P}(\mathcal{P}(V), \mathcal{F}(d)) \), graded \( S(V) \)-module.

\( H^0(\mathcal{P}(V), \mathcal{F}(d)) \)
Consider $G(M): \cdots \rightarrow \hat{E}(-d) \otimes H^0(P(V), \mathcal{F}(d))$

$$\downarrow$$

$$\hat{E}(-d-1) \otimes H^0(P(V), \mathcal{F}(d+1)).$$

This sequence is exact for some $d \rightarrow \cdots$

We obtain a Tate resolution $T(\mathcal{F})$

(First $d$ where exactness holds, consider image and take projective resolution)

**Theorem (EFS):**

$$T^p(\mathcal{F}) = \bigoplus_{i=0}^{\dim P(V)} E(l-p) \otimes H^i(P(V), \mathcal{F}(p-i))$$

- **Ex 1:** $N \cong k$ over $E$

\[\begin{array}{c}
\hat{E}(-n) \otimes \Lambda^n V^* \\
\Lambda^n V \otimes V^* \\
\uparrow
\end{array}\]

This is a Tate resolution $T(\mathcal{O}_{P(V)})$, under

- In fact: $D^b(\text{coh}(P(V))) \rightarrow \text{Tate}(E)$
Ex: \( p \in P^2 \), \( J_p \subset O_{P^2} \).

Cohomology table:

\[
\begin{array}{cccccccc}
6 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 5 & 9 & 14 \\
\hline
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

\( H^2(P^2, J_p(j-2)) \)

\( H^1(P^2, J_p(j-1)) \)

\( H^0(P^2, J_p(j)) \)

\( T^{-3}(J_p) = \wedge^6 E(5) \oplus \wedge^1 E(4) \).

Remark: \( C(E) \xrightarrow{F} C(S) \xleftarrow{G} \).

\( D^b_{fg}(E) \xrightarrow{F} D^b_{fg}(S) \xleftarrow{G} \text{quotient category} \)

\( E\text{-mod} \cong D^b(\text{con}(P^2)) \)

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\( \text{Tot}_E(E) \).