q-Kostka Polynomials and Garnir Translates

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Abstract

We provide an elementary description of the cohomology rings of Springer fibres and derive the positivity property of the q-Kostka polynomials $K_{\lambda\mu}(q)$, following Garsia-Procesi [1] and Bergeron-Garsia [2]. A brief motivational discussion of the cohomology of Springer fibres, Springer representations and certain zero dimensional schemes is included. This exposition was prepared for a talk given in the *Macdonald Polynomials Seminar*, Spring 2013, UC Berkeley, run by Maria Monks and Steven Sam.

1 Introduction

Fix $n \geq 1$ and let S_n be the symmetric group on n letters. The q-Kostka polynomials $K_{\lambda\mu}(q)$ arise as the $\mathbb{Q}(q)$-coefficients of the change of basis matrix between the bases of $\Lambda_{\mathbb{Q}(q)}(X_n)$ given by the *Schur polynomials* $\{S_\lambda(X_n)\}$ and the *Hall-Littlewood polynomials* $\{P_\lambda(X_n; q)\}$:

$$S_\lambda(X_n) = \sum_\mu K_{\lambda\mu}(q) P_\mu(X_n; q).$$

Here, $\Lambda_{\mathbb{Q}(q)}(X_n)$ is the ring of symmetric functions in n variables $X_n = \{x_1, \dots, x_n\}$ over $\mathbb{Q}(q)$. By letting $q \to 1$ in the above equality we recover the well-known identity

$$S_\lambda(X_n) = \sum_\mu K_{\lambda\mu} M_\mu(X_n),$$

where $K_{\lambda\mu}$ are the *Kostka numbers* and $\{M_\lambda(X_n)\}$ are the monomial symmetric polynomials.

In this note we will prove the following property of the $K_{\lambda\mu}(q)$

$$\boxed{K_{\lambda\mu}(q) \in \mathbb{N}(q), \text{ for each } \lambda, \mu; \quad (\textbf{positivity})}$$

To be precise, we will show that

$$\tilde{K}_{\lambda\mu}(q) \stackrel{def}{=} K_{\lambda\mu}(q^{-1}) q^{n(\mu)} \in \mathbb{N}[q]$$

so that the positivity of $K_{\lambda\mu}(q)$ is immediate.

Steven Karp discussed a combinatorial proof of this property due to Lascoux-Schutzenberger making use of the *(co)charge statistic* $c(T)$. Our approach will rely on a representation theoretic interpretation of the $\tilde{K}_{\lambda\mu}(q)$ so that the coefficients of $\tilde{K}_{\lambda\mu}(q)$ arise as the (q-graded) number of times the irreducible S_n-module χ^λ appears in a certain graded S_n-module R_μ.

An outline of the talk is as follows:

1) *Motivation*: we will briefly review the ($\mathbb{Q}$-)cohomology rings of the flag variety of $\mathrm{GL}_n(\mathbb{C})$ and Springer fibres and their realisation as S_n-modules. We will highlight the relationship between these cohomology rings and the coordinate rings of certain 0-dimensional subschemes of $Mat_n(\mathbb{Q})$.

2) *Graded S_n-modules*: we define several graded S_n-modules that are obtained via the standard S_n action on $\mathbb{Q}[x_1, \dots, x_n]$.

3) *Graded characters*: we obtain the graded character of these graded S_n-modules and subsequently derive the positivity property of $K_{\lambda\mu}(q)$.

2 Motivation

The **flag variety** is the projective variety consisting of complete flags in $\mathbb{C}^n$

$$\mathcal{F} = \{(0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n) \mid \dim F_i = i\}$$

Given a nilpotent matrix $A \in Mat_n(\mathbb{C})$, we define the **Springer fibre over** A to be the closed subvariety of flags fixed by A

$$\mathcal{F}_A = \{(F_i) \in \mathcal{F} \mid A(F_i) \subset F_{i-1}\}$$

.

The Springer fibre over A is dependent only upon the conjugacy class of A; since conjugacy classes of nilpotent matrices correspond to partitions of n, we will write $\mathcal{F}_\lambda$ for the Springer fibre of any nilpotent matrix whose Jordan type is λ.

Example 2.1. i) $\mathcal{F}_0 = \mathcal{F}$,

ii) $\mathcal{F}_{(n-1,1)}$ is a union of $(n-1)$ $\mathbb{P}^1_{\mathbb{C}}$'s, with intersection diagram A_{n-1}.

iii) $\mathcal{F}_{(2,2)}$ is the union of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and a $\mathbb{P}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$, intersecting at a point.

Theorem 2.2. *a) There is an isomorphism of graded $\mathbb{Q}$-algebras*

$$\mathbb{Q}[x_1, \dots, x_n]/(e_1(X_n), \dots, e_n(X_n)) \simeq H^*(\mathcal{F}, \mathbb{Q}) \ ; \ x_j \mapsto c_1(L_j)$$

where $L_j \to \mathcal{F}$ is a (tautological) line bundle whose fibre over (F_i) is F_j/F_{j-1}, $e_i(X_n)$ are the elementary symmetric polynomials. We assume here that $H^{2i}(\mathcal{F}, \mathbb{Q})$ is the degree i homogeneous part of $H^*(\mathcal{F}, \mathbb{Q})$

b) (Springer, Hotta-Springer) For each partition λ of n, $H^*(\mathcal{F}_\lambda, \mathbb{Q})$ carries an S_n-action such that the restriction maps

$$H^*(\mathcal{F}, \mathbb{Q}) \to H^*(\mathcal{F}_\lambda, \mathbb{Q}),$$

are surjective and S_n-invariant. $H^{top}(\mathcal{F}_\lambda, \mathbb{Q})$ is the irreducible S_n-module with character χ^λ.

c) *(de Concini-Procesi) For a partition λ of n, define $A_\lambda \stackrel{def}{=} \mathbb{Q}[\overline{\mathcal{O}}_\lambda \cap D_n]$, where $\mathcal{O}_\lambda$ is the orbit of nilpotent matrices of Jordan type λ. There is an S_n-equivariant graded isomorphism*

$$H^*(\mathcal{F}_\lambda, \mathbb{Q}) \simeq A_{\lambda'}$$

where we consider $A_{\lambda'}$ as a quotient of $\mathbb{Q}[x_1, \dots, x_n]$ (the x_i's are the diagonal entries of a matrix) and the S_n-action is induced from permutation of variables.

Example 2.3. We have

$$H^*(\mathcal{F}_{(n-1,1)}, \mathbb{Q}) = H^0(\mathcal{F}_{(n-1,1)}, \mathbb{Q}) \oplus H^2(\mathcal{F}_{(n-1,1)}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^{n-1},$$

and $H^2(\mathcal{F}_{(n-1,1)}, \mathbb{Q})$ is the standard representation.

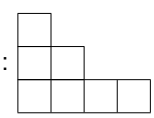
The remaining sections will provide an elementary description of the graded S_n-modules A_λ, for λ a partition of n, and will not mention cohomology, varieties, nilpotent orbits etc. However, it can be useful to keep the geometric picture at hand when considering some of the constructions we encounter in the following sections; indeed, many of the combinatorial arguments arose from the geometric picture (eg see [2], Remark 2.2).

It should be noted that explicit description of the modules A_λ was given first by de Concini-Procesi and Tanisaki, following work of Kraft. The description that we give is independent of their results (ie, it does not rely on their work).

3 Graded Modules, Garnir Translates

Note: in the remaining sections we will adopt the French-style of diagrams/tableau and denote a partition of n by $\lambda = (0 \leq \lambda_1 \leq \dots \leq \lambda_n)$, and the number of nonzero components of λ by $h(\lambda)$. For example, for the partition $\lambda = (0, 0, 0, 0, 1, 2, 4)$ we have $h(\lambda) = 3$ and its Young diagram is

$Y(\lambda)$:



Denote $R = \mathbb{Q}[x_1, \dots, x_n] = \sum_{m \geq 0} R^{(m)}$, considered as a S_n-module in the usual way. There exists an S_n-invariant perfect pairing

$$\langle , \rangle : R \times R \to \mathbb{Q} \; ; \; (P, Q) \mapsto ev_0 \left(P(\partial) Q(X) \right),$$

where $P(\partial) = P(\partial_1, \dots, \partial_n)$, $\partial_i = \frac{\partial}{\partial x_i}$, ev_0 is 'evaluation at $\underline{0}$'.

We have the following basic property: *let $(f_1, \dots, f_k) \subset R$ be a homogeneous ideal. Then,*

$$(f_1, \dots, f_n)^\perp = S(f_1, \dots, f_k) = \{P \in R \mid f_i(\partial) P = 0\}.$$

For the remainder of this section we will fix a partition $\mu = (\mu_1, \dots, \mu_n)$ of n. We now introduce our main objects of study, **the graded S_n-modules H_μ, R_μ.**

Define a collection of homogeneous polynomials C_μ as follows: if $\mu' = (\mu'_1, \dots, \mu'_n)$ is the dual partition, define

$$d_k(\mu) = \mu'_1 + \dots + \mu'_k, \quad k = 1, \dots, n,$$

so that $d_k(\mu)$ is the number of cells of $Y(\mu)$ stictly to the right of column $n-k$ in the $n \times n$ grid containing $Y(\mu)$, with $(0,0)$ at the top right corner, and (n,n) at the bottom left corner (an Arabic grid), so that $Y(\mu)$ is situated in the bottom left corner. For example, if $\mu = (0,0,0,0,1,2,4)$ then $(d_k(\mu)) = (0,0,0,0,1,3,7)$. Then, define

$$C_\mu \stackrel{def}{=} \{e_r(S) \mid S \subset X_n,\ |S| = k > n - \mu_n,\ k \geq r > k - d_k(\mu)\}$$

where $e_i(S)$ is the ith elementary symmetric polynomial in the alphabet S. For example,

$$C_{(1^n)} = \{e_1(X_n), \dots, e_n(X_n)\},$$

$$C_{(1,n-1)} = \{e_r(S) \mid S \subset X_n, n > |S| > 1, |S| \geq r > 1\} \cup \{e_1(X_n), \dots, e_n(X_n)\}$$

In general, $C_\mu \supset C_{(1^n)}$, for any μ.

Define $I_\mu = \langle C_\mu \rangle$, the ideal generated by C_μ; it is a homogeneous, S_n-submodule of R. We now define

$$H_\mu = I_\mu^\perp = \{P \in R \mid e_r(S)(\partial)P = 0,\ \forall e_r(S) \in C_\mu\}, \quad R_\mu = R/I_\mu.$$

Since $\langle , \rangle$ is S_n-invariant, there is an isomorphism of graded S_n-modules

$$H_\mu \simeq R_\mu.$$

If we write

$$\lambda \leq \mu \Longleftrightarrow \lambda_1 + \dots + \lambda_k \leq \mu_1 + \dots + \mu_k, \text{ for each } k, \quad (\textbf{BEWARE!})$$

then

$$\lambda \leq \mu \implies H_\lambda \subset H_\mu.$$

In particular, $H_\mu \subset H_{(1^n)}$, for each μ.

Remark

i) The rings R_μ and H_μ are isomorphic (as graded S_n-modules) to $A_{\mu'}$. Hence, they are isomorphic to $H^*(\mathcal{F}_\mu, \mathbb{Q})$ as graded S_n-mdoules.

ii) The above nesting property of the H_μ should be considered as analogous to the surjectivity result of Springer-Hotta in cohomology of Springer fibres above.

iii) Observe that $H_{(1^n)}$ consists of polynomials annhilated by symmetric homogeneous differential operators: this space is sometimes called the space of *harmonic polynomials*.

We will now identify certain bases of the S_n-modules R_μ, H_μ.

Let T be an injective tableau. Define the **Garnir polynomial** $\Delta_T(X) \in R$ as follows: denote the filled columns of T by $C_1, \dots, C_n$, reading from right to left (possibly some empty columns), and denote the entry in the jth row of C_i(read from the bottom) by C_{ji}. Then,

$$\Delta(C_i) \stackrel{def}{=} \Delta(x_{C_{1i}}, \dots, x_{C_{\mu'_i,i}}),$$

where $\Delta(S)$ is the Vandermonde determinant on the (ordered) alphabet S; define $\Delta(\varnothing) = 1$. Then,

$$\Delta_T(X) = \Delta(C_n)\Delta(C_{n-1}) \cdots \Delta(C_1).$$

A **Garnir translate** is a polynomial

$$\Delta_T(X+t), \quad \text{where } t=(t_1,\dots,t_n)\in\mathbb{Q}^n$$

For example, consider the tableau

$$T: \begin{array}{|c|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 & 7 \\ \hline \end{array}.$$

Then,

$$\Delta_T(x) = \det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & x_3 \\ 1 & x_6 \end{bmatrix}$$

By Taylor's Theorem, together with the observation that $\partial_i = \sum_{r\geq 1}(-1)^r\frac{\Delta_i^r}{r}$, where Δ_i is the difference operator $e^{\partial_i}-1$, we can show that

$$\mathbb{Q}\{\Delta_T(X+t) \mid t\in\mathbb{Q}^n,\ T\in ST(\mu)\} = \mathbb{Q}\{\partial^p\Delta_T(X) \mid T\in ST(\mu)\}\subset R.$$

We will define a basis of H_μ as a certain subset of $\{\partial^p\Delta_T(X) \mid T\in ST(\mu)\}$.

First, we introduce a basis of R_μ: for each $i=1,\dots,h(\mu)$, define the partition $\mu^{(i)}$ of $n-1$ to be the partition obtained by removing the rightmost corner of $Y(\mu)$ in or above row i. Define the set of monomials $B(\mu)$ recursively,

$$B(\mu) = \sum_{i=1}^{h(\mu)} x_n^{i-1}B(\mu^{(i)}), \quad B(1)=1,$$

where '$\sum$' means disjoint union. For example, consider the partition $\mu=(1,3)$ of 4. Then,

$$\mu^{(1)}=(1,2),\ \mu^{(2)}=(3),$$

$$(1,2)^{(1)}=(1^2),\ (1,2)^{(2)}=(2), \quad (3)^{(1)}=(2),$$
$$(1^2)^{(1)}=(1^2)^{(2)}=(1), \quad (2)^1=(1).$$

Hence, $B(1,3)=\{1,x_2,x_3,x_4\}$. In general, we have $B(1,n-1)=\{1,x_2,\dots,x_n\}$. You can check that $B(2,2)=\{1,x_2,x_3,x_4,x_2x_4,x_3x_4\}$.

The set of monomials $B(\mu)$ is the lower ideal (in terms of lattice theory, and with respect to standard partial ordering of monomials) of monomials with maximal elements the monomials

$$m_T(X) = \prod_{i=1}^{h(\mu)} x_i^{h_i(T)-1}, \quad T\in ST(\mu),\ h_i(T)=\text{ height of } i \text{ in } T.$$

To each monomial $X^\epsilon\in B(\mu)$ we associate a unique standard tableau T_ϵ such that $X^\epsilon\leq m_{T_\epsilon}(X)$; say the lex least tableau T such that $X^\epsilon\leq m_T(X)$. Hence, for each $X^\epsilon\in B(\mu)$ we have

$$X^\epsilon = m_{T_\epsilon}(X)/X^{\rho_\epsilon}$$

Theorem 3.1 ([2]). *Let μ be a partition of n.*

a) For each $e_r(S)\in C_\mu$, $e_r(S)(\partial)\Delta_T(X)=0$, so that $\mathbb{Q}\{\partial^p\Delta_T(X)\}\subset H_\mu$.

b) $B(\mu) \subset R_\mu$ *is a basis;* $\{\partial^{\rho_\epsilon}\Delta_{T_\epsilon}(X) \mid X^\epsilon \in B(\mu)\} \subset H_\mu$ *is a basis.*

c) $\dim R_\mu = \dim H_\mu = \binom{n}{\mu} = \frac{n!}{\mu_1!\cdots\mu_n!}$.

d) the top degree homogeneous piece of H_μ *is the irreducible* S_n*-module* χ^μ*, with basis* $\{m_T(x) \mid T \in ST(\mu)\}$.

Idea of proof: Show that $B(\mu)$ spans R_μ; show that $\{\partial^{\rho_\epsilon}\Delta_{T_\epsilon}(X) \mid X^\epsilon \in B(\mu)\}$ is linearly independent in H_μ; use a) and dimension argument to obtain the result. The identification of χ^μ in H_μ is a consequence of the following

Theorem 3.2 (Garnir)**.** *The linear span* $\mathbb{Q}\{\Delta_T(X) \mid T \in ST(\mu)\}$ *is an irreducible* S_n*-module and isomorphic to* χ^μ.

4 Graded Characters

For a graded S_n-module $V = \sum_r V^{(r)}$, we define the **graded character of** V to be

$$\chi^V(q) = \sum_r q^r \chi^{V^{(r)}}.$$

We denote the graded character of R_μ (equiv. H_μ) by $\chi^\mu(q)$.

Example 4.1. i) Let $\mu = (1,3)$. Then, we have that $B(\mu) = \{1, x_2, x_3, x_4\}$ is a basis of R_μ and

$$\chi^\mu(q) = \chi_0 + q\chi^{std}.$$

This follows from the Theorem above or can be computed explicitly: for the degree 1 component we have

Cycle type	1^4	$1^2 2$	13	2^2	4
Matrix class rep.	id	$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\-1&0&0\\-1&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\1&0&-1\\0&1&-1\end{bmatrix}$
Character value	3	1	0	-1	-1

ii) The previous example generalises to

$$\chi^{(1,n-1)}(q) = \chi_0 + q\chi^{std}.$$

Now, each homogeneous piece $R_\mu^{(r)}$ of R_μ is an S_n-module with character

$$\chi^{R_\mu^{(r)}} = \sum_\lambda c_{\lambda\mu}^{(r)} \chi^\lambda.$$

Hence, we can write

$$\chi^\mu(q) = \sum_\lambda c_{\lambda\mu}(q)\chi^\lambda$$

Theorem 4.2 ([1]). *Let μ, λ be partitions of n. Then,*

$$c_{\lambda\mu}(q) = \tilde{K}_{\lambda\mu}(q) = K_{\lambda\mu}(q^{-1})q^{n(\mu)},$$

where $n(\mu) = \sum_{i=0}^{n-1} i\mu_{n-i}$ is the top degree of R_μ.

Corollary 4.3. *i) $K_{\lambda\mu}(q) \in \mathbb{N}[q]$, for any λ, μ.*

ii) $K_{\lambda\mu}(q) << K_{\lambda\nu}(q)$, for $\mu \leq \nu$, where '$<<$' denotes coefficient-wise inequality.

Idea of proof of Theorem: Define

$$C_\mu(X;q) = \sum_\lambda S_\lambda(X)c_{\lambda\mu}(q), \quad \tilde{H}_\mu(X;q) = \sum_\lambda S_\lambda(X)\tilde{K}_{\lambda\mu}(q).$$

Garsia-Procesi show that C_μ and H_μ behave in the same way with respect to the Hall dual of multiplication by elementary symmetric monomials; that is, if Γ_λ denotes the adjoint of $\cdot S_\lambda$ with respect to the Hall inner product, then

$$\langle e_\nu e_\lambda, C_\mu\rangle_H = \langle e_\nu, \Gamma_\lambda C_\mu\rangle_H = \langle e_\nu, \Gamma_\lambda \tilde{H}_\mu\rangle_H = \langle e_\nu e_\lambda, \tilde{H}_\mu\rangle_H$$

so that $C_\mu = \tilde{H}_\mu$ since $\{e_\lambda\}$ generate $\Lambda_{\mathbb{Q}(q)}$, and the result follows.

Idea of proof of Corollary: i) the coefficient of q^i in $\tilde{K}_{\lambda\mu}(q)$ is $\langle \chi^\lambda, \chi^{R_\mu^{(i)}}\rangle$, the multiplicity of χ^λ in $\chi^{R_\mu^{(i)}}$.

ii) is a consequence of the observation that $B(\mu) \subset B(\nu)$ whenever $\mu \leq \nu$, so that $\chi^\mu(q)$ is a subcharacter of $\chi^\nu(q)$.

References

[1] A.M. Garsia, C. Procesi; *On Certain Graded S_n-modules and the q-Kostka Polynomials*, Adv. in Math. **94**, p. 82-138 (1992)

[2] N. Bergeron, A.M. Garsia; *On Certain Spaces of Harmonic Polynomials*, Contemp. Math. **138**, p. 51-86 (1992)