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THE LOCAL STRUCTURE OF POISSON MANIFOLDS

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Historical Introduction

The classical Poisson bracket operation defined on functions on $\mathbf{R}^{2n}$ is

$$(*) \qquad \{f, g\} = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right).$$

In the early nineteenth century, Poisson noticed that the vanishing of $\{f, g\}$ and $\{f, h\}$ imply that of $\{f, \{g, h\}\}$; almost thirty years later Jacobi discovered the identity $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$ which "explains" Poisson's theorem. In his study of general composition laws satisfying the Jacobi identity, Lie [29] defined in local coordinate form what is now known as a Poisson structure. On $\mathbf{R}^r$ such a structure is given by functions $w_{ij}(x_1, \cdots, x_r)$ satisfying the identities

$$w_{ij} + w_{ji} = 0,$$

$$\sum_{l=1}^{r} \left(w_{lj} \frac{\partial w_{ik}}{\partial x_l} + w_{li} \frac{\partial w_{kj}}{\partial x_l} + w_{lk} \frac{\partial w_{ji}}{\partial x_l} \right) = 0,$$

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which imply that the bilinear operation

$$\{F, G\} = \sum_{i,j=1}^{r} w_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}$$

is antisymmetric and satisfies the Jacobi identity; i.e., the algebra of functions $C^\infty(\mathbf{R}^r)$ becomes a Lie algebra. An abstractly defined Lie algebra structure $\{\,,\}$ on $C^\infty(\mathbf{R}^r)$ arises in this way if and only if it satisfies the Leibniz identity $\{F, GH\} = \{F, G\}H + G\{F, H\}$, and this enables us to define a Poisson structure on a manifold P to be a Lie algebra structure $\{\,,\}$ on $C^\infty(P)$ which satisfies the Leibniz identity. The functions w_{ij} may then be seen as the components in local coordinates of an antisymmetric contravariant 2-tensor w; the Jacobi identity may be interpreted as the vanishing on w of a certain natural quadratic differential operator of first order. (Berezin [5], Hermann [19], Lichnerowicz [28], Tulczejew [44]).

Poisson structures have recently become interesting in connection with completely integrable systems and for the hamiltonian formulation of field theoreis in physical variables. (See the papers in Tabor and Treve [43] and references therein.) The aim of this paper is to develop the theory of Poisson manifolds with an eye toward these applications and also a new application—the study of singular limits of hamiltonian systems.

The natural setting for hamiltonian systems is on symplectic manifolds. These may be described as manifolds carrying a Poisson structure which is locally isomorphic to the standard one on an $\mathbf{R}^{2n}$; the coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ are then called *canonical variables*. (There is an extension of this definition to infinite dimensions, which we ignore for now.) A criterion for a Poisson structure to be symplectic is that the tensor w (or the matrix (w_{ij})) have rank everywhere equal to the dimension of the manifold.

The simplest example of a nonsymplectic Poisson structure is $\mathbf{R}^{2n+s}$ with variables $(q_1, \cdots, q_n, p_1, \cdots, p_n, c_1, \cdots, c_s)$ and Poisson brackets given by the standard formula $(*)$. Functions of $(c_1, \cdots, c_s)$ have zero Poisson bracket with everything and are called by Lie *distinguished functions*. (Nowadays they are called invariants or *Casimir functions*.) Each manifold on which all c_j are constant inherits a Poisson structure which is then nondegenerate, so $\mathbf{R}^{2n+s}$ may be thought of as foliated by symplectic manifolds. This turns out to be the situation at the generic points of any Poisson manifold: Lie proved that any Poisson structure on $\mathbf{R}^{2n+s}$ for which the rank of w_{ij} is constant[1] and equal to $2n$ admits s independent distinguished functions and so is locally isomorphic

[1]Actually, Lie assumed the constancy of rank without saying it explicitly.

to the standard example just described; furthermore, the rank of any Poisson structure is constant (and even) on the open subset where it attains its maximal value.[2]

Another important example of a Poisson structure, also introduced by Lie, arises when we are given the structure constants of a Lie algebra, i.e., r^3 constants c_{ijk} satisfying the identities

$$c_{ijk} + c_{jik} = 0, \quad c_{ijj} + c_{jki} + c_{kij} = 0.$$

The functions $w_{ij}(x) = \Sigma_{k=1}^{r} c_{ijk} x_k$ then define a Poisson structure on $\mathbf{R}^r$.[3] The rank of (w_{ij}) is no longer constant (e.g. it is zero at $x = 0$), so Lie's theory applies only at the "regular points" where the rank is maximal. This was enough for Lie to prove his "third theorem" on the existence of a local group with given structure constants and to classify locally the r'tuples $(\phi_1, \cdots, \phi_r)$ of functions on $\mathbf{R}^{2n}$ satisfying the bracket relations $\{\phi_i, \phi_j\} = \Sigma_{k=1}^{r} c_{ijk} \phi_k$. In the course of this classification, Lie observed that the leaves of the symplectic foliation of $\mathbf{R}^r$ were just the "smallest invariant submanifolds for the dual of the adjoint group" acting on $\mathbf{R}^r$.

Nowadays we think of $\mathbf{R}^r$ as the dual space $\mathfrak{g}^*$ of the Lie algebra $(\mathfrak{g}, [\,,])$ whose basis $X_1, \cdots, X_k$ satisfies $[X_i, X_j] = \Sigma_{k=1}^{r} c_{ijk} X_k$. The "dual of the adjoint group" is called the coadjoint representation, and the smallest invariant manifolds are the coadjoint orbits.[4] Their symplectic structure was rediscovered in the 1960's by Kirillov [24], Kostant [26], and Souriau [41], whose arguments covered the singular coadjoint orbits as well as the regular ones. In fact it turns out (see Kirillov [25]) that through each point of every Poisson manifold there passes a symplectic manifold whose dimension equals the rank of the Poisson structure there, and the Poisson structure is built up out of the Poisson structures on these *symplectic leaves*. For a $\mathfrak{g}^*$ carrying Lie's Poisson structure, the symplectic leaves are just the coadjoint orbits.

This property of being smoothly decomposed into symplectic manifolds of different dimensions seems to make Poisson manifolds an appropriate setting for studying a phenomenon which is quite common in mechanics: if a mechanical system is modeled by a symplectic manifold, then when a parameter in the system reaches a limiting value (usually 0 or ∞), the limiting system also has a symplectic formulation, but with fewer degrees of freedom. Examples of this are:

(i) the restricted 3-body problem in celestial mechanics (mass $\to 0$);

[2] These results were rediscovered by Lichnerowicz [28] and Hermann [20].

[3] Introduced again by Berezin [4].

[4] By the way, Lie also discovered and used, more or less as it is used now, the momentum mapping and its equivariance under the coadjoint representation.

(ii) the guiding center limit for a particle in an electromagnetic field (charge/mass $\to \infty$);

(iii) the limit of discrete vortices in the motion of incompressible fluids (concentration of vorticity $\to \infty$);

(iv) the classical limit of quantum mechanics ($\hbar \to 0$).

There are also examples when the number of degrees of freedom remains the same but the global structure of a symplectic manifold of group changes, as in the newtonian limit of special relativity ($c \to \infty$); these too should be accessible to study in terms of Poisson structures.

A good part of this paper then will consist of an extension of Lie's results to the case of variable rank. Most of the methods are ones which Lie himself would have used, but the geometric language developed since his time gives us new insight and enables us to say some things more efficiently.

There is one new aspect to the theory which is peculiar to the case of variable rank. At a point where the functions $w_{ij}(x)$ vanish, we can linearize them to obtain a linear Poisson structure living on the tangent space. The *linearization question* is whether the original Poisson structure is locally equivalent to a linear one. The question has different answers according to whether we work in the formal, analytic, or C^∞ categories, and there remain some unsolved problems.

Some interesting work on the singular structure of Poisson manifolds, from a more algebraic viewpoint than ours, can be found in the papers of Vinogradov and Krasilshchik [46] and Berger [6].

To end this introduction, the author wishes to point out that a great deal of interest today in symplectic actions and coadjoint orbits lies in their relevance for understanding unitary actions of Lie groups. This is truly a modern development for which there does not seem to be any precedent in Lie's work.

Acknowledgments

Bob Hermann and Wilfrid Schmid, in conversations and their published work (Hermann [20], Schmid [40]) pointed the author toward Lie's [29] pioneering work on Poisson manifolds. Hans Duistermaat was responsible for my recent education in the geometry of Lie groups, some of which found direct applications in this paper. The author's interest in applications of Poisson manifolds to the understanding of variational principles and Clebsch variables was stimulated at a conference at the La Jolla Institute (December 1981) organized by Michael Tabor and Yvain Treve. Conversations there with Frank

Henyey, Darryl Holm, John Hubbard, Boris Kupershmidt, and Robert Littlejohn were especially helpful. Finally, Jerry Marsden has been a patient listener and frequent collaborator in the work described here—our joint work described in Marsden and Weinstein [32] and Marsden, Ratiu and Weinstein [33] is a close companion to the present paper.

1. Poisson manifolds and mappings

A *Poisson structure* on a manifold P is defined as a Lie algebra structure $\{\,,\}$ on $C^\infty(P)$ satisfying the Liebniz identity $\{FG, H\} = F\{G, H\} + \{F, H\}G$. The bracket operation $\{\,,\}$ is thus a derivation in each entry, and so in particular for each function H there is a vector field ξ_H such that $\xi_H \cdot F = \{F, H\}$ for all F. ξ_H is called the *hamiltonian vector field* generated by H. At any point $p \in P$, the value of $\{F, H\}$ and hence of ξ_H depends only on the differential of H, so there is a bundle map $B : T^*P \to TP$ such that $\xi_H = B \circ dH$ for all H. We may also think of B as defining a contravariant antisymmetric 2-tensor w in P, for which $\{F, G\} = \langle (dF, dG), w \rangle$. The tensor w is sometimes called a *cosymplectic structure*; the Jacobi identity for $\{\,,\}$ is equivalent to the vanishing of the so-called Schouten [42] bracket $[w, w]$.

In local coordinates $(x_1,\cdots,x_r)$, a Poisson structure is determined by the component functions $w_{ij}(x)$ of w. In terms of the bracket we have simply $\{x_i, x_j\} = w_{ij}(x)$; in other words, the Poisson structure is specified if we give the bracket relations satisfied by the coordinate functions. This is exactly how Lie thought of Poisson structures; he noted that the bracket

$$\{F, G\} = \sum_{i,j=1}^{r} w_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}$$

satisfies the asymmetry and Jacobi conditions if and only if $w_{ij} = -w_{ji}$ and

$$\sum_{l=1}^{r} \left(w_{lj} \frac{\partial w_{ik}}{\partial x_l} + w_{li} \frac{\partial w_{kj}}{\partial x_l} + w_{lk} \frac{\partial w_{jl}}{\partial x_l} \right) = 0.$$

It follows from the Jacobi identity that the (local) flow of each ξ_H preserves the Poisson structure, and also that $\xi_{\{H,K\}} = \xi_K \xi_H - \xi_H \xi_K = [\xi_H, \xi_K]$; we use Arnold's [3] sign convention for the bracket of vector fields. Thus the Poisson structure determines a Lie algebra homomorphism from $C^\infty(P)$ to the infinitesimal automorphisms of the Poisson structure.

Automorphisms of a Poisson manifold are an example of *Poisson mappings* defined in general as maps $J : P_1 \to P_2$ between Poisson manifolds such that $\{F \circ J, G \circ J\}_1 = \{F, G\}_2 \circ J$, or equivalently by the condition $J_* w_1(x) = w_2(J(x))$. A *Poisson submanfold* is a submanifold Q in a Poisson manifold P

with a Poisson structure for which the inclusion is a Poisson mapping. Such a structure, if it exists, is unique.

Lemma 1.1. *$Q \subseteq P$ is a Poisson submanifold if and only if each tangent space T_xQ contains the image of $B_x : T_x^*P \to T_xP$, i.e., if and only if all hamiltonian vector fields are tangent to Q.*

Proof. Given functions $\overline{F}$ and $\overline{G}$ on Q, extend them to functions F and G on P. (It will be enough to do this locally.) The tangency condition implies that the restriction of $\{F, G\}$ to Q depends only on $\overline{F}$ and $\overline{G}$, so there is an induced bracket operation on Q which is easily seen to make Q into a Poisson submanifold. If the tangency condition fails, the bracket of extended functions depends upon the extensions, so the restriction map $C^\infty(P) \to C^\infty(Q)$ cannot define a homomorphism for any Lie algebra structure on $C^\infty(Q)$.

A related fact is

Lemma 1.2. *Let $J : P_1 \to P_2$ be a Poisson mapping, and H a function on P_2. Then the trajectories on P of the hamiltonian vector field ξ_H are the projections under J of the trajectories on P_2 of $\xi_{H \circ J}$.*

Proof. Let $\sigma(t)$ be an integral curve of $\xi_{H \circ J}$. Then for any function G on P_2,

$$\begin{aligned}\frac{d}{dt}(G \circ (J \circ \sigma)) = \frac{d}{dt}((G \circ J) \circ \sigma) &= \{G \circ J, H \circ J\} \circ \sigma \\ &= (\{G, H\} \circ J) \circ \sigma = \{G, H\} \cdot (J \circ \sigma).\end{aligned}$$

It follows that $J \circ \sigma$ is an integral curve of ξ_H. Clearly, all integral curves of ξ_H arise in this way. q.e.d.

See Guillemin and Sternberg [15] and Mishchenko and Fomenko [34] for applications of this lemma to the study of collective and invariant motion.

The *rank* of a Poisson structure at a point $x \in P$ is defined to be the rank of $B_x : T_x^*P \to T_xP$. (In local coordinates it is the rank of the matrix $w_{ij}(x)$.) The invariance of the Poisson structure under hamiltonian flows implies the constancy of rank along the orbits of such flows. From this it is not hard to derive the following result, due to Kirillov [25] (see also Hermann [17]) in general but to Lie [29] for the case of constant rank: Every Poisson manifold is essentially a union of symplectic manifolds which fit together in a smooth way.

Proposition 1.3. *Define a relation $\sim$ on P by declaring $x \sim y$ if y can be reached from x by a piecewise smooth curve, each segment of which is a trajectory of a hamiltonian vector field. Then $\sim$ is an equivalence relation, and the equivalence classes are Poisson submanifolds of P. The bracket of functions on P is therefore determined by the brackets of their restrictions to these submanifolds. The dimension of each such submanifold Q is equal to the rank of the Poisson structure (of P or of Q) at each point of Q.*

A Poisson structure for which the rank is everywhere equal to the dimension of the manifold is called *nondegenerate* or *symplectic*. The symplectic structure on such a manifold is the 2-form Ω defined by $\Omega(\xi, \eta) = w(B^{-1}\xi, B^{-1}\eta)$; the usual Poisson brackets and hamiltonian vector fields for this symplectic structure are just the ones for the Poisson structure. We shall call the equivalence classes Q in Proposition 1.3 the *symplectic leaves* of P, since they form a foliation everywhere that the rank of the Poisson structure is locally constant (for instance on the open dense subset of points with locally maximal rank).

A *Casimir function*, or *invariant*, on a Poisson manifold is a function C such that $\{C, F\} = 0$ for all functions F; equivalent conditions are that C is constant along the orbits of all hamiltonian vector fields or that the hamiltonian vector field of C itself is zero. A Casimir function is constant along each symplectic leaf, and in a region where the rank is constant, the symplectic leaves are exactly the common level manifolds of the local Casimir functions.

It is sometimes possible to define an induced Poisson structure on a submanifold which is not a Poisson submanifold. For instance, only Poisson submanifolds of a symplectic manifold are its connected components, but any submanifold on which the symplectic form is nondegenerate has an induced symplectic, hence Poisson structure. The following proposition will be useful in the next section.

Proposition 1.4. *Let Q be a submanifold of the Poisson manifold P such that the following conditions are satisfied at each $x \in Q$:*
(i) $B_x(T_xQ^\perp) \cap T_xQ = \{0\}$,
(ii) $T_xQ^\perp \cap \operatorname{Ker} B_x = \{0\}$,
where $T_xQ^\perp$ is the annihilator of T_xQ in T_x^*P. Then there is a naturally induced Poisson structure on Q.

Proof. Condition (ii) is equivalent to $T_xQ + \operatorname{Im} B_x = T_xP$, i.e., Q intersects each symplectic leaf transversely. Thus Q is a union of manifolds, each of which is a submanifold of a symplectic manifold. Now the intersection $B_x(T_xQ^\perp) \cap T_xQ$ is just the null space of the induced symplectic form on $\operatorname{Im} B_x \cap T_xQ$, so condition (i) tells us that the intersection manifolds making up Q are in fact symplectic. Thus we can put together the Poisson brackets on these manifolds to get a bracket on Q. To see that the resulting structure is smooth, we note that conditions (i) and (ii) imply that $B_x(T_xQ^\perp) \oplus T_xQ = T_xP$ for each Q, giving a smooth bundle projection π from the restricted tangent bundle T_QP onto TQ. The induced Poisson structure on Q is then defined by the composed map $T^*Q \xrightarrow{\pi^*} T_Q^*P \xrightarrow{B_Q} T_QP \xrightarrow{\pi} TQ$. (The argument involving submanifolds was needed to show that the induced structure satisfies the Jacobi and Leibniz identities.)

2. Splitting

The product $P_1 \times P_2$ becomes a Poisson manifold in an obvious way so that the projections $\pi_j: P_1 \times P_2 \to P_j$ are Poisson mappings, and $\pi_1(C^\infty(P_1))$ and $\pi_2(C^\infty(P_2))$ are commuting subalgebras of $C^\infty(P)$. In terms of coordinates, if bracket relations $\{x_i, x_j\} = w_{ij}(x)$ and $\{y_i, y_j\} = v_{ij}(y)$ are given, then these define a bracket on functions of x and y when augmented by the relations $\{x_i, y_j\} = 0$.

The main theorem of this section states that every Poisson manifold is locally the product of a symplectic manifold and a Poisson manifold having a point where the rank is zero.

Theorem 2.1 (*Splitting theorem*). *Let x_0 be any point in a Poisson manifold P. Then there are a neighborhood U of x_0 in P and an isomorphism $\phi = \phi_S \times \phi_N$ from U to a product $S \times N$ such that S is symplectic and the rank of N at $\phi_N(x_0)$ is zero. The factors S and N are unique up to local isomorphism.*

Proof. (The following existence proof for the decomposition is essentially a standard proof of Darboux's theorem for symplectic manifolds.) If the rank of P at x_0 is zero, we are done; otherwise, there are functions q_1' and p_1 such that $\{q_1', p_1\}(x_0) \neq 0$. Then $\xi_{p_1}(x_0) \neq 0$, so by straightening out this vector field near x_0 we can find a function q_1 such that $\xi_{p_1}q_1 = 1$, i.e., $\{q_1, p_1\} = 1$. Now the vector fields ξ_{q_1} and ξ_{p_1} commute, so we can find functions $x_3,\cdots,x_n$ such that the x_j's commute with q_1 and p_1, and $(q_1, p_1, x_3,\cdots,x_r)$ form a coordinate system. By Poisson's theorem, the brackets $\{x_i, x_j\}$ commute with q_1 and p_1 as well; thus

$$0 = \{\{x_i, x_j\}, q_1\} = \frac{\partial\{x_i, x_j\}}{\partial p_1}\{p_1, q_1\} + \text{zeros},$$

so $\{x_i, x_j\}$ is independent of p_1 and similarly of q_1. Hence we have relations $\{x_i, x_j\} = v_{ij}(x)$, so P is locally the product of a two-dimensional symplectic manifold and a Poisson manifold whose dimension and rank at each point are two less than for P.

Repeating this process as often as necessary, we find "canonical" coordinates $(q_1,\cdots,q_k, p_1,\cdots,p_k, y_1,\cdots,y_s)$ with $\{q_i, q_j\} = \{p_i, p_j\} = \{q_i, y_i\} = \{p_i, y_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$, and $\{y_i, y_j\} = v_{ij}(y)$ with $v_{ij} = 0$ at x_0. This completes the existence proof.

The uniqueness of the symplectic factor S follows from the fact that its dimension is just the rank of P at x_0. A "natural" representative for S is the symplectic leaf through x_0. (In fact, the argument up to this point gives another proof of the existence of the symplectic leaf.)

To prove the uniqueness of N, we will identify the functions $v_{ij}(y)$ as the induced Poisson structure on a submanifold transverse to the symplectic leaf, using Proposition 1.4.

Suppose then that we have coordinates (p, q, y) as just constructed. We may assume that all the coordinates vanish at x_0, so we may identify the symplectic factor S with the set $y = 0$ and the singular factor N with $q = p = 0$. We shall show that the Poisson structure on N is just the induced structure from Proposition 1.4, with N considered as a submanifold.

The tangent bundle TN is spanned by the vector fields $\partial/\partial y_i$ and the orthogonal bundle $TQ^{\perp}$ by the 1-forms dq_j and dp_j. Now $B(dq_j) = -\partial/\partial p_j$ and $B(dp_j) = \partial/\partial q_j$, so the hypothesis of Proposition 1.4 are satisfied. The projection π from $T_N P$ to TN kills $\partial/\partial q_j$ and $\partial/\partial p_j$ and fixes $\partial/\partial y_i$, so π^* takes dy_i to dy_i. Thus the induced Poisson structure maps

$$dy_i \mapsto dy_i \mapsto \sum_{j=1}^{s} v_{ij}\frac{\partial}{\partial y_j} \mapsto \sum_{j=1}^{s} v_{ij}\frac{\partial}{\partial y_j}$$

and is equal to the structure on the factor N.

Since the induced structure on a submanifold is canonically defined, our proof of the uniqueness theorem will be complete if we can prove the following lemma.

Lemma 2.2. *Let N_0 and N_1 be submanifolds of the Poisson manifold P having complementary dimension to a symplectic leaf S. Suppose that each N_i intersects S at a single point transversely. Then there is an automorphism of P which maps a neighborhood of $N_0 \cap S$ in N_0 onto a neighborhood of $N_1 \cap S$ in N_1. (This automorphism induces an isomorphism of the induced Poisson structures on the neighborhoods.)*

Proof. By integrating hamiltonian vector fields, we can map $N_0 \cap S$ to $N_1 \cap S$, so we may assume that these points of intersection are actually the same. Next we interpolate between N_0 and N_1 by a family of manifolds which can be defined by equations of the form $q_j = Q_j(y_1, \cdots, y_s, t)$ and $p_j = P_j(y_1, \cdots, y_s, t)$, where $t \in [0, 1]$, and P_j and Q_j are smooth functions.

We will find a family Ψ_t of Poisson automorphisms such that Ψ_0 is the identity, and Ψ_t maps a neighborhood of $N_0 \cap S$ in N_0 onto N_t. The Ψ_t are obtained by integrating a time-dependent hamiltonian vector field ξ_{H_t}. In order that the integral curves of ξ_{H_t} "track" the N_t's, we must have the equations

$$\dot{q}_j = \sum_{i=1}^{s} \frac{\partial Q_j}{\partial y_i}\dot{y}_i + \frac{\partial Q_j}{\partial t},$$

$$\dot{p}_j = \sum_{i=1}^{s} \frac{\partial P_j}{\partial y_i}\dot{y}_i + \frac{\partial P_j}{\partial t}$$

satisfied along N_t. Substituting the coefficients of ξ_{H_t} for $\dot{q}_j$, $\dot{p}_j$ and $\dot{y}_i$, we find

$$\frac{\partial H_t}{\partial p_j} = \sum_{i,l=1}^{s} \frac{\partial Q_j}{\partial y_i}\frac{\partial H_t}{\partial y_l}v_{il} + \frac{\partial Q_j}{\partial t},$$

$$-\frac{\partial H_t}{\partial q_j} = \sum_{i,l=1}^{s} \frac{\partial P_j}{\partial y_i}\frac{\partial H_t}{\partial y_l}v_{il} + \frac{\partial P_j}{\partial t}.$$

In other words, the effect on H of vector fields of the form

$$\frac{\partial}{\partial p_j} + \sum c_{jl}\frac{\partial}{\partial y_l}, \quad \frac{\partial}{\partial q_j} + \sum d_{jl}\frac{\partial}{\partial y_l}$$

is prescribed along N_t. On $N_t \cap S$, the functions v_{il} and hence c_{jl} and d_{jl} are zero, so the vector fields are transverse to N_t, and the required function H_t can be found. (For instance, one could begin by setting $H_t = 0$ along N_t.) This completes the proof of the lemma and, hence, of the splitting theorem. q.e.d.

An easy corollary of the existence part of the splitting theorem is the following basic result of Lie [29] mentioned in the Introduction.

Corollary 2.3. *Suppose that the rank of the Poisson manifold is constant near x_0. Then there are coordinates $(q_1,\cdots,q_k,\ p_1,\cdots,p_k,\ y_1,\cdots,y_s)$ near x_0 satisfying the canonical bracket relations $\{q_i, q_j\} = \{p_i, p_j\} = \{q_i, y_j\} = \{p_i, y_j\} = \{y_i, y_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$.*

Proof. In the coordinates given by the splitting theorem, the rank of $v_{ij}(y) = \{y_i, y_j\}$ must be constant. Since it is zero at x_0, it must be identically zero. q.e.d.

As a consequence of the proof of the uniqueness part of the splitting theorem, there is a well-defined notion of "transverse Poisson structure" along any symplectic leaf: the induced Poisson structures on all cross sections to the symplectic leaf are locally isomorphic, but there is no natural representative for this transverse structure.

Another consequence of the splitting theorem is that the symplectic leaves in P near x_0 are locally products of S with symplectic leaves in the transverse Poisson manifold. A global conclusion may be drawn if the transverse Poisson manifold N is *stable* near $N \cap S$ in the sense that N has arbitrarily small neighborhoods of $N \cap S$ which are invariant under all hamiltonian vector