Chapter 4

VECTORS AND FOUNDATIONS

1 Algebraic operations with vectors

119. Definition of vectors. In physics, some quantities (e.g. distances, volumes, temperatures, or masses) are completely characterized by their magnitudes, expressed with respect to a chosen unit by real numbers. These quantities are called *scalars.* Some others (e.g. velocities, accelerations, or forces) cannot be characterized only by their magnitudes, because they may differ also by their directions. Such quantities are called *vectors.*

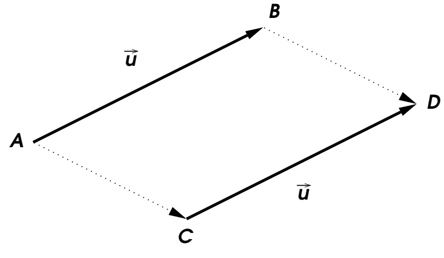


Figure 121

To represent a vector quantity geometrically, we draw an arrow connecting two points in space, e.g. A with B (Figure 121). We call it a **directed segment** with the **tail** A and **head** B, and indicate this in writing as $\overrightarrow{AB}$.

The same **vector** can be represented by different directed segments. By definition, two directed segments ($\overrightarrow{AB}$ and $\overrightarrow{CD}$) **represent the same vector** if they are obtained from each other by translation (§72). In other words, the directed segments must have the same length, lie on the same line or two parallel lines, and point toward the same direction (out of two possible ones). When this is the case, we write $\overrightarrow{AB} = \overrightarrow{CD}$ and say that the vectors represented by these directed segments are **equal**. Note that $\overrightarrow{AB} = \overrightarrow{CD}$ exactly when the quadrilateral $ABDC$ is a parallelogram.

We will also denote a vector by a single lower case letter with an arrow over it, e.g. the vector $\vec{u}$ (Figure 121).

120. Addition of vectors. Given two vectors $\vec{u}$ and $\vec{v}$, their **sum** $\vec{u} + \vec{v}$ is defined as follows. Pick any directed segment $\overrightarrow{AB}$ (Figure 122) representing the vector $\vec{u}$. Represent the vector $\vec{v}$ by the directed segment $\overrightarrow{BC}$ whose tail B coincides with the head of $\overrightarrow{AB}$. Then the directed segment $\overrightarrow{AC}$ represents the sum $\vec{u} + \vec{v}$.

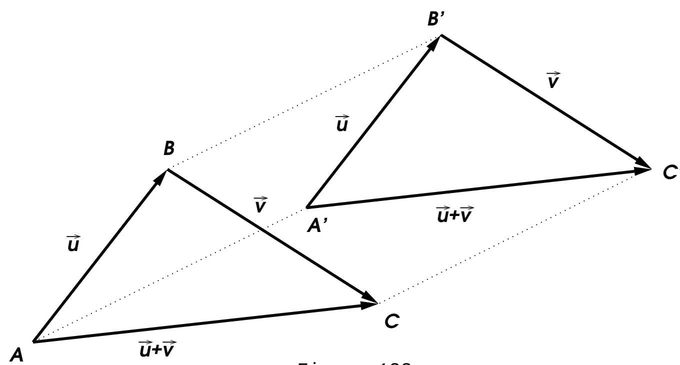


Figure 122

The sum of vectors thus defined does not depend on the choice of the directed segment representing the vector $\vec{u}$. Indeed, if another directed segment $\overrightarrow{A'B'}$ is chosen to represent $\vec{u}$, and respectively the directed segment $\overrightarrow{B'C'}$ (with the tail B') represents $\vec{v}$, then the quadrilaterals $ABB'A'$ and $BCC'B'$ are parallelograms. Therefore the segments AA' and CC' are congruent, parallel, and have the same direction (since they are congruent to, parallel to, and have the same direction as BB'), and hence the quadrilateral $ACC'A'$ is a parallelogram too. Thus the directed segments $\overrightarrow{AC}$ and $\overrightarrow{A'C'}$ represent the same vector.

Addition of vectors is **commutative**, i.e. the sum does not depend on the order of the summands:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \text{ for all vectors } \vec{u} \text{ and } \vec{v}.$$

Indeed, represent the vectors by directed segments $\overrightarrow{AB}$ and $\overrightarrow{AD}$ with the same tail A (Figure 123). In the plane of the triangle ABD, draw lines $BC \| AD$ and $DC \| AB$, and denote by C their intersection point. Then $ABCD$ is a parallelogram, and hence $\overrightarrow{DC} = \vec{u}$, $\overrightarrow{BC} = \vec{v}$. Therefore the diagonal $\overrightarrow{AC}$ of the parallelogram is a directed segment representing both $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$.

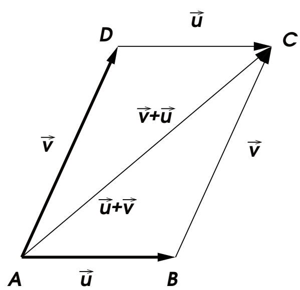


Figure 123

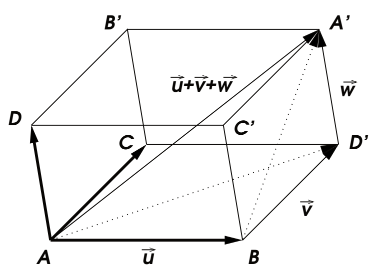


Figure 124

Addition of vectors is **associative**, i.e. the sum of three (or more) vectors does not depend on the order in which the additions are performed:

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \text{ for all vectors } \vec{u}, \vec{v}, \text{ and } \vec{w}.$$

Indeed, represent the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ by directed segments $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ with the same tail A (Figure 124), and then construct the parallelepiped $ABCDA'B'C'D'$ all of whose edges are parallel to AB, AC, or AD. Then the diagonal directed segment $\overrightarrow{AA'}$ represents the sum $\vec{u} + \vec{v} + \vec{w}$ regardless of the ordering of the summands or the order in which the additions are performed. For instance,

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + (\overrightarrow{BD'} + \overrightarrow{D'A'}) = \vec{u} + (\vec{v} + \vec{w}),$$

since $\overrightarrow{BD'} = \overrightarrow{AC} = \vec{v}$ and $\overrightarrow{D'A'} = \overrightarrow{AD} = \vec{w}$. At the same time,

$$\overrightarrow{AA'} = \overrightarrow{AD'} + \overrightarrow{D'A'} = (\overrightarrow{AB} + \overrightarrow{BD'}) + \overrightarrow{D'A'} = (\vec{u} + \vec{v}) + \vec{w}.$$

121. Multiplication of vectors by scalars. Given a scalar (i.e. a real number) α and a vector $\vec{u}$, one can form a new vector denoted $\alpha\vec{u}$ and called the **product** of the scalar and the vector.

Namely, represent the vector $\vec{u}$ by any directed segment $\overrightarrow{AB}$ (Figure 125) and apply to it the homothety (see §§70–72) with the coefficient $\alpha \neq 0$ with respect to any center S. Then the resulting directed segment $\overrightarrow{A'B'}$ represents the vector $\alpha\vec{u}$. In other words, since the triangles SAB and $SA'B'$ are similar, the directed segment $\overrightarrow{A'B'}$ representing the vector $\alpha\vec{u}$ is parallel to $\overrightarrow{AB}$ (or lies on the same line), is $|\alpha|$ times longer than $\overrightarrow{AB}$, and has the same direction as $\overrightarrow{AB}$ when α is positive, and the opposite direction when α is negative.

We will often call vectors $\vec{u}$ and $\alpha\vec{u}$ **proportional** and refer to the number α as the coefficient of proportionality.

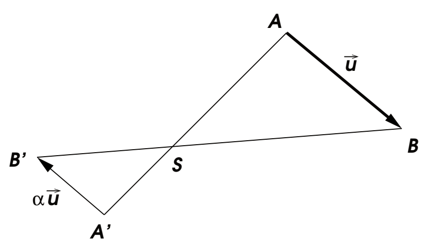


Figure 125

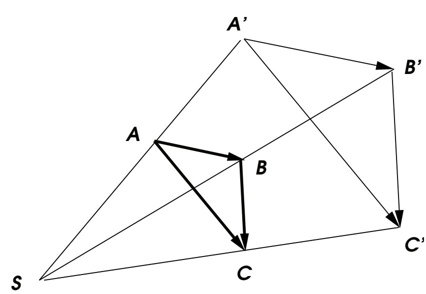


Figure 126

In the special case of $\alpha = 0$, the product $0\vec{u}$ is represented by any directed segment $\overrightarrow{SS}$ whose tail and head coincide. The corresponding vector is called the **zero vector** and is denoted by $\vec{0}$. Thus

$$0\vec{u} = \vec{0} \quad \text{for every vector } \vec{u}.$$

Multiplication by scalars is **distributive** *with respect to addition of vectors,* i.e. for all vectors $\vec{u}$ and $\vec{v}$ and every scalar α we have:

$$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}.$$

Indeed, let the sides of the triangle ABC (Figure 126) represent respectively: $\overrightarrow{AB}$ the vector $\vec{u}$, $\overrightarrow{BC}$ the vector $\vec{v}$, and $\overrightarrow{AC}$ their sum $\vec{u} + \vec{v}$, and let $\triangle A'B'C'$ be homothetic to $\triangle ABC$ (with respect to any center S) with the homothety coefficient α. Then

$$\overrightarrow{A'B'} = \alpha\vec{u}, \quad \overrightarrow{B'C'} = \alpha\vec{v}, \quad \text{and} \quad \overrightarrow{A'C'} = \alpha(\vec{u} + \vec{v}).$$

Since $\overrightarrow{A'C'} = \overrightarrow{A'B'} + \overrightarrow{B'C'}$, the distributivity law follows.

Two more properties[1] of multiplication of vectors by scalars:

$$(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}, \quad \text{and} \quad (\alpha\beta)\vec{u} = \alpha(\beta\vec{u})$$

follow from the geometric meaning of operations with numbers. Indeed, let $\overrightarrow{OU} = \vec{u}$ (Figure 127). The infinite line OU can be identified with the number line (see Book I, §153) by taking the segment OU for the unit of length and letting the points O and U represent the numbers 0 and 1 respectively. Then any scalar α is represented on this number line by a unique point A such that $\overrightarrow{OA} = \alpha\overrightarrow{OU}$. Furthermore, addition of vectors on the line and their multiplication by scalars is expressed as addition and multiplication of the corresponding numbers. For example, if B is another point on the line such that $\overrightarrow{OB} = \beta\overrightarrow{OU}$, then the vector sum $\overrightarrow{OA}+\overrightarrow{OB}$ corresponds to the number $\alpha+\beta$ on the number line, i.e. $\alpha\overrightarrow{OU}+\beta\overrightarrow{OU} = (\alpha+\beta)\overrightarrow{OU}$. Similarly, multiplying by a scalar α the vector $\overrightarrow{OB}$ (corresponding to β on the number line) we obtain a new vector corresponding to the product $\alpha\beta$ of the numbers, i.e. $\alpha(\beta\overrightarrow{OU}) = (\alpha\beta)\overrightarrow{OU}$.

`Examples.` (1) If a vector $\vec{u}$ is represented by a directed segment $\overrightarrow{AB}$, then the opposite directed segment $\overrightarrow{BA}$ represents the vector $(-1)\vec{u}$, also denoted simply by $-\vec{u}$. Note that **opposite vectors** add up to $\vec{0}$. This is obvious from $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0}$, but also follows formally from the distributivity law:

$$\vec{0} = 0\vec{u} = (-1+1)\vec{u} = (-1)\vec{u} + 1\vec{u} = -\vec{u} + \vec{u}.$$

(2) If vectors $\vec{u}$ and $\vec{v}$ are represented by the directed segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with a common tail, then the directed segment $\overrightarrow{BC}$ connecting the heads represents the **difference** $\vec{v} - \vec{u}$ (because $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$).

In applications of vector algebra to geometry, it is often convenient to represent all vectors by directed segments with a common tail, called the **origin**, which can be chosen arbitrarily. Once an origin O is chosen, each point A in space becomes represented by a unique vector, $\overrightarrow{OA}$, called the **radius-vector** of the point A with respect to the origin O.

[1] They are called: **distributivity** with respect to addition of scalars, and **associativity** with respect to multiplication by scalars.

122. `Problem.` *Compute the radius-vector $\vec{m}$ of the barycenter of a triangle ABC* (Figure 128), *given the radius-vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ of its vertices.*

Recall that the barycenter is the intersection point of the medians. Denote A' the midpoint of the side BC, and on the median AA' mark the point M which divides the median in the proportion $AM : MA' = 2 : 1$. We have: $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{BC} = \vec{c} - \vec{b}$, and hence

$$\overrightarrow{AA'} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \vec{b} - \vec{a} + \frac{1}{2}\vec{c} - \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{b} + \vec{c} - 2\vec{a}).$$

Therefore

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AA'} = \vec{a} + \frac{1}{3}(\vec{b} + \vec{c} - 2\vec{a}) = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c}).$$

Clearly, the same result remains true for each of the other two medians. Thus the head of the radius-vector

$$\vec{m} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$$

lies on all the three medians and thus coincides with the barycenter. As a by-product, we have obtained a new proof of the concurrency theorem (Book I, §142): *the three medians of a triangle meet at the point dividing each of them in the proportion* $2 : 1$ *counting from the vertex.*

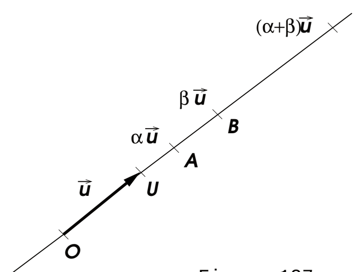


Figure 127

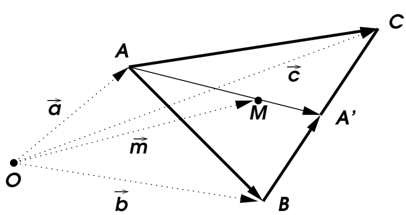


Figure 128

123. The dot product. Given two vectors $\vec{u}$ and $\vec{v}$, their **dot product** $\vec{u} \cdot \vec{v}$ (also known as **scalar product**) is a *number* defined as the product of the lengths of the vectors and the cosine of the angle between their directions. Thus, if the vectors are represented by the directed segments $\overrightarrow{OU}$ and $\overrightarrow{OV}$ (Figure 129), then

$$\vec{u} \cdot \vec{v} = OU \cdot OV \cdot \cos \angle VOU.$$

In particular, the dot product of a vector with itself is equal to the square of the vector's length:

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2, \quad \text{or} \quad |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}.$$

If $\theta(\vec{u}, \vec{v})$ denotes the angle between the directions of two non-zero vectors, then

$$\mathbf{cos}\, \theta(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}.$$

Thus, the dot product operation captures information about distances and angles.

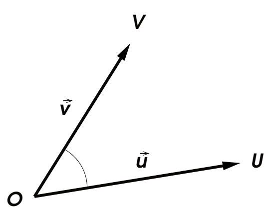


Figure 129

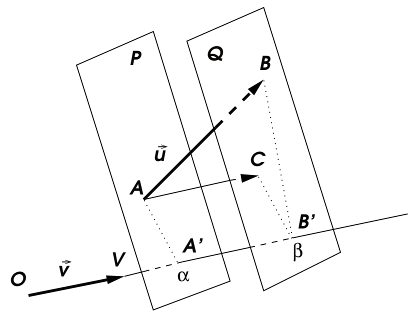


Figure 130

Now let us define the **signed projection** of any vector $\vec{u}$ to the direction of a **unit vector** $\vec{v}$, i.e. assuming that $|\vec{v}| = 1$. Let directed segments $\overrightarrow{AB}$ and $\overrightarrow{OV}$ (Figure 130) represent the two vectors. Consider projections of the points A and B to the line OV. For this, draw through A and B the planes P and Q perpendicular to the line OV until they intersect it at the points A' and B'. Identifying the line OV with the number line, we represent the positions of these points by numbers α and β, and introduce the signed projection as their difference $\beta - \alpha$. It does not depend on the choice of directed segments representing the vectors because it is equal to their dot product: $\beta - \alpha = \vec{u}\, \mathbf{cos}\, \theta(\vec{u}, \vec{v})$. Indeed, draw through the point A the line $AC \| OU$, and extend it to the intersection point C with the plane Q. Then $AC = A'B'$ (as segments of parallel lines between two parallel planes), and $\angle BAC = \theta(\vec{u}, \vec{v})$. The sign of $\beta - \alpha$ is positive if the direction of $\overrightarrow{A'B'}$ agrees with the direction of $\overrightarrow{OV}$, i.e. whenever $\angle BAC$ is acute, and negative otherwise, i.e. when the angle is obtuse. We find therefore that $\beta - \alpha = AB \cdot \mathbf{cos}\, \angle BAC = \vec{u} \cdot \vec{v}$.

We obtain the following geometric interpretation of the dot product operation: *the dot product of any vector with a unit vector is equal to the signed projection of the former to the direction of the latter.*

124. Algebraic properties of the dot product.

(1) *The dot product operation is* **symmetric**:

$$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u} \quad \text{for all vectors } \vec{u} \text{ and } \vec{v},$$

because obviously $\mathbf{cos}\,\theta(\vec{u},\vec{v})=\mathbf{cos}\,\theta(\vec{v},\vec{u})$.

(2) *The dot product $\vec{u}\cdot\vec{v}$ is* **homogeneous** *(of degree 1) with respect to either vector, i.e. for all vectors $\vec{u}$, $\vec{v}$, and any scalar α*

$$(\alpha\vec{u})\cdot\vec{v}=\alpha(\vec{u}\cdot\vec{v})=\vec{u}\cdot(\alpha\vec{v}).$$

It suffices to verify the first equality only (as the second one follows from it due to the symmetricity of the dot product). The length of $\alpha\vec{u}$ is $|\alpha|$ times greater than the length of $\vec{u}$. Therefore the property is obvious for positive (or zero) α since in this case the directions of these vectors coincide. In the case of negative α, the vectors $\vec{u}$ and $\alpha\vec{u}$ have opposite directions (Figure 131). Then the angles they make with the vector $\vec{v}$ are supplementary, so that their cosines are opposite, and the equality remains true.

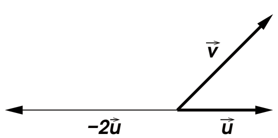


Figure 131

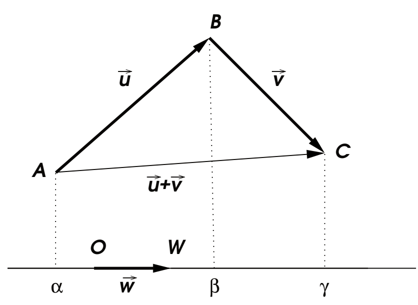


Figure 132

(3) *The dot product is* **additive** *with respect to each of the vectors:*

$$(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w} \quad\text{and}\quad \vec{w}\cdot(\vec{u}+\vec{v})=\vec{w}\cdot\vec{u}+\vec{w}\cdot\vec{v}$$

for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. Due to symmetricity, it suffices to verify only the first equality. We may assume that $\vec{w}\neq\vec{0}$ (because otherwise all three terms vanish). Due to homogeneity, dividing each term by the length of the vector $\vec{w}$, one reduces the equality to its special case when $|\vec{w}| = 1$. Let ABC (Figure 132) be a triangle such that $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{BC} = \vec{v}$, and hence $\overrightarrow{AC} = \vec{u} + \vec{v}$. Denote by A', B', and C' the projections of the points A, B, and C to the line of the unit directed segment $\overrightarrow{OW} = \vec{w}$. Considering this line as the number line, represent the projection points by numbers α, β, and γ. Then

$$\vec{u} \cdot \vec{w} = \beta - \alpha, \quad \vec{v} \cdot \vec{w} = \gamma - \beta, \quad \text{and} \quad (\vec{u} + \vec{v}) \cdot \vec{w} = \gamma - \alpha.$$

Since $(\beta - \alpha) + (\gamma - \beta) = \gamma - \alpha$, the required equality holds.

125. `Examples.` Some applications of the dot product operation are based on the simplicity of its algebraic properties.

(1) *Perpendicular vectors have zero dot product* because $\cos 90^\circ = 0$. Therefore, if we denote by $\vec{u}$ and $\vec{v}$ the legs $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a right triangle ABC, then its hypotenuse $\overrightarrow{AC}$ is $\vec{u} + \vec{v}$, and the square of its length is computed as follows:

$$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot (\vec{u} + \vec{v}) + \vec{v} \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v},$$

since $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$. Thus $AC^2 = AB^2 + BC^2$, and so we have re-proved the Pythagorean theorem once again.

(2) More generally, given any triangle ABC, put $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$ and compute:

$$\begin{aligned} BC^2 &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) = \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \\ &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v} = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos \angle BAC. \end{aligned}$$

This is the law of cosines (Book I, §205).

EXERCISES

232. Prove that for every closed broken line $ABCDE$,

$$\overrightarrow{AB} + \overrightarrow{BC} + \cdots + \overrightarrow{DE} + \overrightarrow{EA} = \vec{0}.$$

233. Prove that if the sum of three unit vectors is equal to $\vec{0}$, then the angle between each pair of these vectors is equal to 120°.

234. Prove that if four unit vectors lying in the same plane add up to $\vec{0}$ then they form two pairs of opposite vectors. Does this remain true if the vectors do not have to lie in the same plane?

235.*** Let $ABCDE$ be a regular polygon with the center O. Prove that

$$\overrightarrow{OA} + \overrightarrow{OB} + \cdots + \overrightarrow{OE} = \vec{0}.$$

236. Along three circles lying in the same plane, vertices of a triangle are moving clockwise with the equal constant angular velocities. Find out how the barycenter of the triangle is moving.

237. Prove that if AA' is a median in a triangle ABC, then

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$

238. Prove that from segments congruent to the medians of a given triangle, another triangle can be formed.

239. Sides of one triangle are parallel to the medians of another. Prove that the medians of the latter triangle are parallel to the sides of the former one.

240. From medians of a given triangle, a new triangle is formed, and from its medians, yet another triangle is formed. Prove that the third triangle is similar to the first one, and find the coefficient of similarity.

241. Midpoints of AB and CD, and of BC and DE are connected by two segments, whose midpoints are also connected. Prove that the resulting segment is parallel to AE and congruent to $\frac{1}{4}AE$.

242. Prove that a point X lies on the line AB if and only if for some scalar α and any origin O the radius-vectors satisfy the equation:

$$\overrightarrow{OX} = \alpha\overrightarrow{OA} + (1-\alpha)\overrightarrow{OB}.$$

243. Prove that if the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular, then $|\vec{u}| = |\vec{v}|$.

244. For arbitrary vectors $\vec{u}$ and $\vec{v}$, verify the equality:

$$|\vec{u}+\vec{v}|^2 + |\vec{u}-\vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2,$$

and derive the theorem: the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

245. Prove that for every triangle ABC and every point X in space, $\overrightarrow{XA}\cdot\overrightarrow{BC} + \overrightarrow{XB}\cdot\overrightarrow{CA} + \overrightarrow{XC}\cdot\overrightarrow{AB} = 0$.

246.* For four arbitrary points A, B, C, and D in space, prove that if the lines AC and BD are perpendicular, then $AB^2 + CD^2 = BC^2 + DA^2$, and *vice versa*.

247. Given a quadrilateral with perpendicular diagonals, show that every quadrilateral, whose sides are respectively congruent to the sides of the given one, has perpendicular diagonals.

248. A regular triangle ABC is inscribed into a circle of radius R. Prove that for every point X of this circle, $XA^2+XB^2+XC^2 = 6R^2$.

249.* Let $A_1B_1A_2B_2\dots A_nB_n$ be a $2n$-gon inscribed into a circle. Prove that the length of the vector $\overrightarrow{A_1B_1}+\overrightarrow{A_2B_2}+\cdots+\overrightarrow{A_nB_n}$ does not exceed the diameter.
`Hint:` Consider projections of the vertices to any line.

250.* A polyhedron is filled with air under pressure. The pressure force to each face is the vector perpendicular to the face, proportional to the area of the face, and directed to the exterior of the polyhedron. Prove that the sum of these vectors equals $\vec{0}$.
`Hint:` Take the dot-product with an arbitrary unit vector, and use Corollary 2 of §65.

2 Applications of vectors to geometry

126. `Theorem.` ***If the circumcenter*** (O, Figure 133) ***of a triangle*** (ABC) ***is chosen for the origin, then the radius-vector of the orthocenter is equal to the sum of the radius-vectors of the vertices.***

Denote the radius-vectors of the vertices A, B, and C by $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively. Then $|\vec{a}|=|\vec{b}|=|\vec{c}|$, since O is the circumcenter. Let H be the point in the plane of the triangle such that $\overrightarrow{OH}=\vec{a}+\vec{b}+\vec{c}$. It is required to show that H is the orthocenter. Compute the dot product $\overrightarrow{CH}\cdot\overrightarrow{AB}$. Since $\overrightarrow{CH}=\overrightarrow{OH}-\overrightarrow{OC}=(\vec{a}+\vec{b}+\vec{c})-\vec{c}=\vec{a}+\vec{b}$, and $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}$, we find:

$$\overrightarrow{CH}\cdot\overrightarrow{AB}=(\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})=\vec{b}\cdot\vec{b}-\vec{a}\cdot\vec{a}=|\vec{b}|^2-|\vec{a}|^2=0.$$

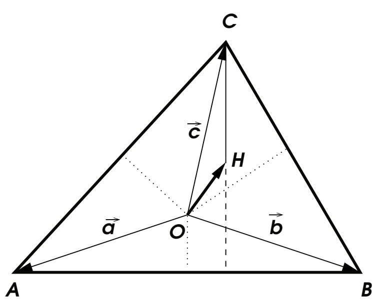


Figure 133

Vanishing of the dot product of two vectors means that these vectors are perpendicular (unless one of them is zero, in which case the angle is, strictly speaking, ill-defined). We conclude that the line CH is perpendicular to AB (unless H and C coincide), i.e. (in either case) the point H lies on the altitude dropped from the vertex C to the side AB, or on the extension of this altitude. Since the same applies to each of the other two altitudes, it follows that the three altitudes, or their extensions, pass through the point H.

`Corollaries.` (1) We have obtained a new proof of the theorem (Book I, §141): *Altitudes of a triangle are concurrent.*

(2) *In every triangle, the circumcenter O, barycenter M, and orthocenter H are collinear. More precisely, M divides the segment OH in the proportion $OM : MH = 1 : 2$.* Indeed, according to §122, we have:

$$\overrightarrow{OM} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c}) = \frac{1}{3}\overrightarrow{OH}.$$

`Remarks.` (1) The segment containing the circumcenter, barycenter, and orthocenter is called **Euler's line** of the triangle (see also Book I, Exercises 226–228).

(2) In the previous theorem, operations with vectors allow one to formulate new results (and re-prove old ones) about familiar notions of plane geometry. In the next example, vectors turn out to be useful although the formulation of the problem does not involve any vectors at all. In such situations, in order to apply vector algebra, points of interest can be represented by their radius-vectors. If it is unclear from the context which of the given points should play the role of the origin, it is advisable (although not necessary) to avoid making any artificial choice. Instead, one can chose an arbitrary point not mentioned in the problem — the resulting conclusion will not depend on this choice.

127. `Problem.` *Given a triangle ABC* (Figure 134), *a new triangle $A'B'C'$ is drawn in such a way that A' is centrally symmetric to A with respect to the center B, B' is centrally symmetric to B with respect to the center C, C' is centrally symmetric to C with respect to the center A, and then the triangle ABC is erased. Reconstruct $\triangle ABC$ from $\triangle A'B'C'$ by straightedge and compass.*

Pick an arbitrary point O as the origin, and denote by $\vec{a}$, $\vec{a'}$, $\vec{b}$, etc. the radius-vectors of the points A, A', B, etc. If two points are centrally symmetric with respect to a center, then the radius-vector of the center is equal to the average of the radius-vectors of the points. Therefore, from the hypotheses of the problem, we have:

$$\vec{b} = \frac{1}{2}(\vec{a} + \vec{a'}), \quad \vec{c} = \frac{1}{2}(\vec{b} + \vec{b'}), \quad \vec{a} = \frac{1}{2}(\vec{c} + \vec{c'}).$$

Assuming that $\vec{a}'$, $\vec{b}'$, and $\vec{c}'$ are given, solve this system of equations for $\vec{a}$, $\vec{b}$, and $\vec{c}$. For this, replace $\vec{b}$ in the 2nd equation by its expression from the 1st, and substitute the resulting expression for $\vec{c}$ into the 3rd equation. We obtain:

$$\vec{a} = \frac{1}{2}\left(\vec{c}' + \frac{1}{2}\left(\vec{b}' + \frac{1}{2}\left(\vec{a}' + \vec{a}\right)\right)\right) = \frac{1}{2}\vec{c}' + \frac{1}{4}\vec{b}' + \frac{1}{8}\vec{a}' + \frac{1}{8}\vec{a}.$$

Therefore,

$$\frac{7}{8}\vec{a} = \frac{1}{2}\vec{c}' + \frac{1}{4}\vec{b}' + \frac{1}{8}\vec{a}', \quad \text{or} \quad \vec{a} = \frac{1}{7}\vec{a}' + \frac{2}{7}\vec{b}' + \frac{4}{7}\vec{c}'.$$

The directed segment $\overrightarrow{OA}$ representing the last vector expression is not hard to construct by straightedge and compass, starting from given directed segments $\vec{a}' = \overrightarrow{OA'}$, $\vec{b}' = \overrightarrow{OB'}$, and $\vec{c}' = \overrightarrow{OC'}$. The vertices B and C of $\triangle ABC$ can be constructed using the expressions:

$$\vec{b} = \frac{1}{7}\vec{b}' + \frac{2}{7}\vec{c}' + \frac{4}{7}\vec{a}' \quad \text{and} \quad \vec{c} = \frac{1}{7}\vec{c}' + \frac{2}{7}\vec{a}' + \frac{4}{7}\vec{b}'.$$

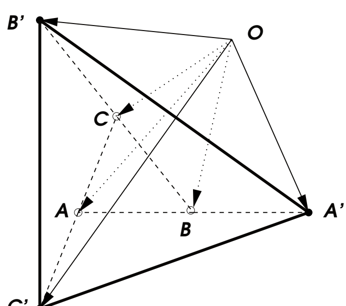


Figure 134

128. The center of mass. By a **material point** we will mean a point in space equipped with a **mass**, which can be any real number. Unless the opposite is specified, we will assume all masses *positive*.[2] The following notion is borrowed from physics.

Given a system of n material points $A_1, A_2, \ldots, A_n$ of masses $m_1, m_2, \ldots, m_n$, their **center of mass** (or **barycenter**) is the material point whose mass m is equal to the total mass of the system:

$$m = m_1 + m_2 + \cdots + m_n,$$

[2]When both positive and negative masses occur, we refer to the latter ones as **pseudo-masses**.

and the position A is determined by the condition:

$$m_1\overrightarrow{AA_1} + m_2\overrightarrow{AA_2} + \cdots + m_n\overrightarrow{AA_n} = \vec{0}.$$

In other words, the above **weighted sum** of the radius-vectors of the material points with respect to the center of mass as the origin is equal to zero.

With respect to an arbitrary origin O, the radius-vector $\vec{a} = \overrightarrow{OA}$ of the center of mass can be computed in terms of the radius-vectors $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$ of the points. We have:

$$\vec{0} = m_1(\vec{a_1} - \vec{a}) + \cdots + m_n(\vec{a_n} - \vec{a}) = m_1\vec{a_1} + \cdots + m_n\vec{a_n} - m\vec{a},$$

and therefore

$$\vec{a} = \frac{1}{m}(m_1\vec{a_1} + m_2\vec{a_2} + \cdots + m_n\vec{a_n}).$$

This formula establishes the existence and the uniqueness of the mass center of any system of n material points (even with negative masses, as long as the total mass m of the system is non-zero).

`Examples.` (1) In a system of two material points, we have: $m_1\overrightarrow{AA_1} + m_2\overrightarrow{AA_2} = \vec{0}$, or equivalently, $\overrightarrow{AA_1} = -\frac{m_2}{m_1}\overrightarrow{AA_2}$. Hence the center of mass A lies on the segment A_1A_2 (Figure 135), connecting the points, and divides it in the proportion $A_1A : AA_2 = m_2 : m_1$ (i.e. the mass center is closer to the point of greater mass).

(2) Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be radius-vectors of three given material points of *equal* mass. Then $\frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$ is the radius-vector of the center of mass. Comparing with §122, we conclude that the center of mass coincides with the barycenter of the triangle with vertices at the three given points.

129. Regrouping. Most applications of centers of mass to geometry rely on their **associativity**, or **regrouping** property.

`Theorem.` ***If a system of material points is divided into two (or more) parts, and then each part is replaced by a single material point representing its center of mass, then the center of mass of the resulting system of two (or more) material points coincides with the center of mass of the original system.***

Say, let A_1, A_2 and A_3, A_4, A_5 (Figure 136) be two parts of a system of five material points with masses $m_1, \dots, m_5$. We are required to show that if A' and A'' are the positions of the centers of mass of these parts, and $m' = m_1 + m_2$ and $m'' = m_3 + m_4 + m_5$ are their respective masses, then the center of mass of this pair of material points coincides (as a material point, i.e. in regard to both its mass and position) with the center of mass of the whole system of five material points.

Firstly, we note that the sum $m' + m''$ indeed coincides with the total mass $m = m_1 + m_2 + \cdots + m_5$ of the whole system. Secondly, using the radius-vectors $\vec{a'}, \vec{a''}, \vec{a_1}, \ldots, \vec{a_5}$ of the points $A', A'', A_1, \ldots, A_5$ with respect to any origin, we find:

$$\vec{a'} = \frac{1}{m'}(m_1\vec{a_1} + m_2\vec{a_2}), \quad \vec{a''} = \frac{1}{m''}(m_3\vec{a_3} + m_4\vec{a_4} + m_5\vec{a_5}).$$

The center of mass of this pair of material points has the radius-vector

$$\frac{1}{m' + m''}(m'\vec{a'} + m''\vec{a''}) = \frac{1}{m}(m_1\vec{a_1} + m_2\vec{a_2} + m_3\vec{a_3} + m_4\vec{a_4} + m_5\vec{a_5}).$$

Thus, it coincides with the radius-vector of the center of mass of the whole system.

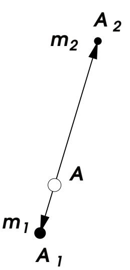


Figure 135

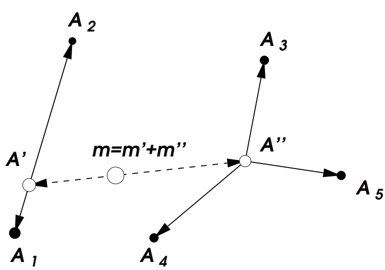


Figure 136

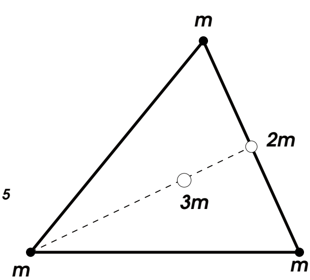


Figure 137

`Example.` Equip each vertex of a given triangle with the same mass m (Figure 137) and compute the center of mass for vertices of one of the sides first. It lies at the midpoint of that side and carries the mass $2m$. By the theorem, the center of mass of the whole system lies on the median connecting this midpoint with the opposite vertex and divides it in the proportion $2m : m$ counting from the vertex. Since the center of mass is the same regardless of the order of grouping, we derive concurrency of medians once again.

130. Ceva's theorem.

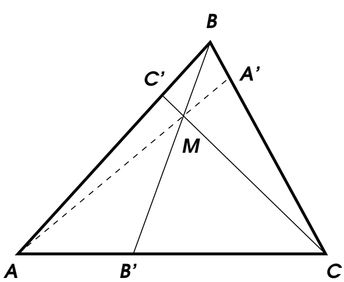
`Theorem.` ***Given a triangle*** ABC (Figure 138) ***and points*** A', B', ***and*** C' ***on the sides*** BC, CA, ***and*** AB ***respectively, the lines*** AA', BB', ***and*** CC' ***are concurrent if and only if the vertices can be equipped with masses such that*** A', B', C' ***become centers of mass of the pairs:*** B ***and*** C, C ***and*** A, A ***and*** B ***respectively.***

Suppose A, B, and C are material points, and A', B' and C' are positions of the centers of mass of the pairs B and C, C and A, A and B. Then, by the regrouping property, the center of mass of the whole system lies on each of the segments AA', BB', and CC'. Therefore these segments are concurrent.

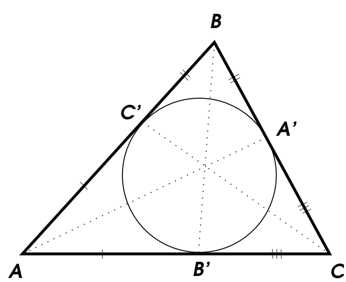
Conversely, assume that the lines AA', BB', and CC' are concurrent. Assign an arbitrary mass $m_A = m$ to the vertex A, and then assign masses to the vertices B and C so that C' and B' become the centers of mass of the pairs A and B, and A and C respectively, namely:

$$m_B = \frac{AC'}{C'B} m, \quad \text{and} \quad m_C = \frac{AB'}{B'C} m.$$

Then the center of mass of the whole system will lie at the intersection point M of the segments BB' and CC'. On the other hand, by regrouping, it must lie on the line connecting the vertex A with the center of mass of the pair B and C. Therefore the center of mass of this pair is located at the intersection point A' of the line AM with the side BC.



`Figure 138`



`Figure 139`

`Corollary` (Ceva's theorem). *In a triangle* ABC, *the segments* AA', BB', *and* CC', *connecting the vertices with points on the opposite sides, are concurrent if and only if*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1. \qquad (*)$$

Indeed, when the lines are concurrent, the equality becomes obvious when rewritten in terms of the masses:

$$\frac{m_B}{m_A}\cdot\frac{m_C}{m_B}\cdot\frac{m_A}{m_C}=1.$$

Conversely, the relation $(*)$ means that if one assigns masses as in the proof of the theorem, i.e. so that $m_B : m_A = AC' : C'B$ and $m_C : m_A = AB' : B'C$, then the proportion $m_C : m_B = BA' : A'C$ holds too. Therefore all three points C', B', and A' are the centers of mass of the corresponding pairs of vertices. Now the concurrency property is guaranteed by the theorem.

`Problem.` *In a triangle ABC* (Figure 139), *let A', B', and C' denote points of tangency of the inscribed circle with the sides. Prove that the lines AA', BB', and CC' are concurrent.*

`Solution 1.` We have: $AB' = AC'$, $BC' = BA'$, and $CA' = CB'$ (as tangent segments drawn from the vertices to the same circle). Therefore the relation $(*)$ holds true, and the concurrency follows from the corollary.

`Solution 2.` Assigning masses $m_A = 1/AB' = 1/AC'$, $m_B = 1/BC' = 1/BA'$, and $m_C = 1/CA' = 1/CB'$, we make A', B', and C' the centers of mass of the corresponding pairs of vertices, and therefore the concurrency follows from the theorem.

131. Menelaus' theorem.

`Lemma.` ***Three points** A_1**,** A_2**, and** A_3 **are** **collinear** (i.e. lie on the same line) **if and only if they can be equipped with non-zero pseudo-masses** m_1**,** m_2**, and** m_3 (they are allowed therefore to have different signs) **such that***

$$m_1+m_2+m_3=0,\quad\text{and}\quad m_1\overrightarrow{OA_1}+m_2\overrightarrow{OA_2}+m_3\overrightarrow{OA_3}=\vec{0}.$$

If the points are collinear, then one can make the middle one (let it be called A_3) the center of mass of the points A_1 and A_2 by assigning their masses according to the proportion $m_2 : m_1 = A_1A_3 : A_3A_2$. Then, for any origin O, we have: $m_1\overrightarrow{OA_1}+m_2\overrightarrow{OA_2}-(m_1+m_2)\overrightarrow{OA_3}=\vec{0}$, i.e. it suffices to put $m_3 = -m_1 - m_2$.

Conversely, if the required pseudo-masses exist, one may assume (changing, if necessary, the signs of all three) that one of them (say, m_3) is negative while the other two are positive. Then $m_3 = -m_1 - m_2$, and the relation $m_1\overrightarrow{OA_1}+m_2\overrightarrow{OA_2}-(m_1+m_2)\overrightarrow{OA_3}=\vec{0}$ means that A_3 is the position of the center of mass of the pair of material points A_1 and A_2. Thus A_3 lies on the segment A_1A_2.

`Corollary` (Menelaus' theorem.) *Any points A', B', and C'* (Figure 140) *lying on the sides BC, CA, and AB respectively of $\triangle ABC$, or on their extensions, are collinear, if and only if*

$$\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=1.$$

`Remark.` This relation looks identical to $(*)$, and it may seem puzzling how the same relation can characterize triples of points A', B', C' satisfying two different geometric conditions. In fact in Menelaus' theorem (see Figure 140), either one or all three of the points must lie on extensions of the sides, so that the same relation is applied to two mutually exclusive geometric situations. Furthermore, let us identify the sides of $\triangle ABC$ with number lines by directing them as shown on Figure 140, i.e. the side AB from A to B, BC from B to C, and CA from C to A. Then the segments AC', $C'B$, BA', etc. in the above relation can be understood as signed quantities, i.e. real numbers whose absolute values are equal to the lengths of the segments, and the signs are determined by the directions of the vectors $\overrightarrow{AC'}$, $\overrightarrow{C'B}$, $\overrightarrow{BA'}$, etc. on the respective number lines. With this convention, the correct form of the relation in Menelaus' theorem is:

$$\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=-1, \qquad (**)$$

thereby differing from the relation in Ceva's theorem by the sign.[3]

To prove Menelaus' theorem in this improved formulation, note that we can always assign to the vertices A, B, and C some real numbers a, b, and c so that C' (resp. B') becomes the center of mass of the pair of points A and B (resp. C and A) equipped with pseudo-masses a and $-b$ (resp. c and $-a$). Namely, it suffices to take: $-b : a = AC' : C'B$ and $-a : c = CB' : B'A$. Then the relation $(**)$ means that $BA' : A'C = -c : b$, i.e. A' is the center of mass of the pair B and C equipped with pseudo-masses b and $-c$ respectively. Thus, we have: $(a-b)\overrightarrow{OC'} = a\overrightarrow{OA} - b\overrightarrow{OB}$, $(c-a)\overrightarrow{OB'} = c\overrightarrow{OC} - a\overrightarrow{OA}$, and $(b-c)\overrightarrow{OA'} = b\overrightarrow{OB} - c\overrightarrow{OC}$. Adding these equalities, and putting $m_A = b-c$, $m_B = c-a$, $m_C = a-b$, we find:

$$m_A\overrightarrow{OA'} + m_B\overrightarrow{OB'} + m_C\overrightarrow{OC'} = \vec{0}, \quad m_A + m_B + m_C = 0.$$

[3] In Ceva's theorem, it is also possible to apply the sign convention and consider points on the extensions of the sides. Then the relation $(*)$ remains the correct criterion for the three lines to be concurrent (or parallel). When $(*)$ holds, an even number (i.e. 0 or 2) of the points lie on the extensions of the sides.

Therefore the points A', B', and C' are collinear.

Conversely, for any points C' and B' in the interior or on the extensions of the sides AB and CA, we can find a point A' on the line BC such that the relation (∗∗) holds true. Then, according to the previous argument, points A', B', and C' are collinear, i.e. point A' must coincide with the point of intersection of the lines $B'C'$ and BC. Thus the relation (∗∗) holds true for any three collinear points on the sides of a triangle or on their extensions.

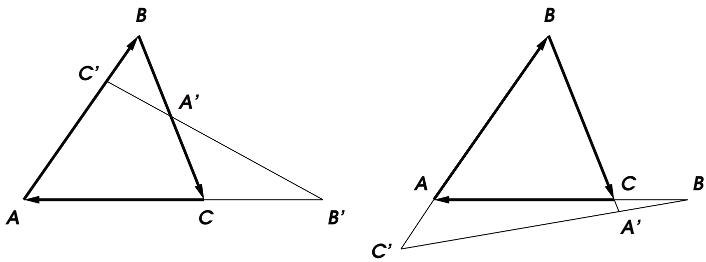


Figure 140

132. The method of barycenters demystified. This method, developed and applied in §§128–131 to some problems of *plane* geometry, can be explained using geometry of vectors in *space*.

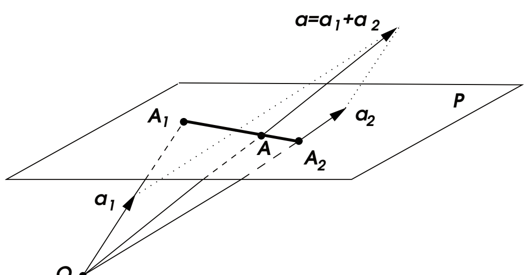


Figure 141

Position the plane P in space in such a way (Figure 141) that it misses the point O chosen for the origin. Then, to each point A on the plane, one can associate a line in space passing through the origin, namely the line OA. When the point comes equipped with a mass (or pseudo-mass) m, we associate to this material point on the plane the vector $\vec{a} = m\overrightarrow{OA}$ in space. We claim that this way, *the center of mass of a system of material points on the plane corresponds to the sum of the vectors associated to them in space.* Indeed, if A denotes the center of mass of a system of n material points $A_1, \ldots, A_n$ in the plane of masses $m_1, \ldots, m_n$, then the total mass is equal to

$m = m_1 + \cdots + m_n$, and the corresponding vector in space is

$$\vec{a} = m\overrightarrow{OM} = m_1\overrightarrow{OA_1} + \cdots + m_n\overrightarrow{OA_n} = \vec{a_1} + \cdots + \vec{a_n}.$$

In particular, the regrouping property of the center of mass follows from associativity of the addition of vectors.

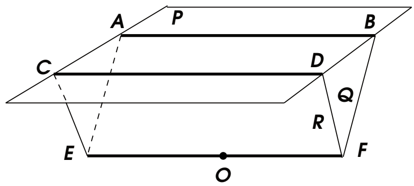


Figure 142

Remark. The above method of associating lines passing through the origin to points of the plane P turns out to be fruitful and leads to the so-called **projective geometry**. In projective geometry, beside ordinary points of the plane P, there exist "points at infinity." They correspond to lines passing through the origin and parallel to P (e.g. EF on Figure 142). Moreover, lines on the plane P (e.g. AB or CD) correspond to planes (Q or R) passing through the origin. When $AB \| CD$, the lines do not intersect on the plane P, but in projective geometry they intersect "at infinity," namely at the "point" corresponding to the line EF of intersection of the planes Q and R. Thus, the optical illusion that two parallel rails of a railroad track meet at the line of the horizon becomes reality in projective geometry.

EXERCISES

251. In the plane, let A, B, C, D, E be arbitrary points. Construct the point O such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE}$.

252.* In a circle, three non-intersecting chords AB, CD, and EF are given, each congruent to the radius of the circle, and the midpoints of the segments BC, DE, and FA are connected. Prove that the resulting triangle is equilateral.

253. Prove that if a polygon has several axes of symmetry, then they are concurrent.

254. Prove that the three segments connecting the midpoints of opposite edges of a tetrahedron bisect each other.

255. Prove that bisectors of exterior angles of a triangle meet extensions of the opposite sides at collinear points.

256. Formulate and prove an analogue of the previous problem for bisectors of one exterior and two interior angles of a triangle.

257. Prove that if vertices of a triangle are equipped with masses proportional to the opposite sides, then the center of mass coincides with the incenter.

258.* Prove that tangents to a circle at the vertices of an inscribed triangle intersect extensions of the opposite sides at collinear points.

259. In the plane, three circles of different radii are given outside each other, and for each pair, the external common tangents are drawn up to their intersection point. Prove that the three intersection points are collinear.

260. In the plane, three pairwise disjoint circles are given outside each other, and for each pair, the intersection point of internal common tangents is constructed. Prove that the three lines, connecting each intersection point with the center of the remaining circle, are concurrent.

261. Prove the following reformulation of Ceva's theorem: On the sides BC, CA, and AB of $\triangle ABC$ (Figure 138), three points A', B', and C' are chosen. Prove that the lines AA', BB', and CC' are concurrent if and only if

$$\frac{\mathbf{sin}\,\angle ACC'}{\mathbf{sin}\,\angle C'CB}\;\frac{\mathbf{sin}\,\angle BAA'}{\mathbf{sin}\,\angle A'AC}\;\frac{\mathbf{sin}\,\angle CBB'}{\mathbf{sin}\,\angle B'BA}=1.$$

262. Give a similar reformulation of Menelaus' theorem.

263. Two triangles ABC and $A'B'C'$ are given in the plane, and through the vertices of each of them, lines parallel to the respective sides of the other are drawn. Prove that if the lines of one of the triples are concurrent, then the lines of the other triple are concurrent too.

264.* Prove **Pappus' theorem**: If points A, B, C lie on one line, and A', B', C' on another, then the three intersection points of the lines AB' and BA', BC' and CB', AC' and CA', are collinear.

`Hint:` Reduce to the case of parallel lines using projective geometry, i.e. by restating the problem about points and lines in the plane in terms of corresponding lines and planes in space.

265.* Prove **Desargues' theorem**: In the plane, if the lines AA', BB', and CC' connecting vertices of two triangles ABC and $A'B'C'$ are concurrent, then the three intersection points of each pair of extended respective sides (i.e. AB and $A'B'$, BC and $B'C'$, and CA and $C'A'$) are collinear, and *vice versa*.

`Hint:` Represent the diagram as a projection from space.