Math 53. Multi-variable Calculus. Problems from past final exams

1. Find and classify all critical points of the function $f(x,y) = y^3 + 3x^2y - 12y$ as local maxima, local minima or saddle points.

2. Find the maximum and minimum values of the function $x + y + z$ on the surface of the ellipsoid

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{6} = 12.$$

3. Change the order of integration in the triple integral:

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\ dz\ dy\ dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\ dy\ dx\ dz\ .$$

4. A homogeneous ball of radius R and mass density μ rotates with the angular velocity ω about a line passing through its center. Compute the kinetic energy accumulated in the ball due to this rotation.

5. An ink spot of initially round shape $(x-1)^2 + y^2 \leq a^2$ is carried by the flow of a two-dimensional fluid with the velocity vector field

$$\vec{V} = (x + e^{-x^2} \cos y)\vec{i} + (2y + 2xe^{-x^2} \sin y)\vec{j}.$$

Find the area of the ink spot at the moment t.

6. Calculate the area of the surface

$$z = a(1 - \frac{2x^2}{a^2} - \frac{2y^2}{a^2}), \quad z \geq 0.$$

7. Compute the flux of the vector field $\vec{F} = \vec{r}/|\vec{r}|^3$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, across the surface $(x+1)^2 = y^2 + z^2,\ x \leq 0$, equipped with an orientation of your choice.

8. Give an example of a divergence-free vector field which is not the curl of any vector field. Justify your answer.

9. Find the domain D on the plane for which the line integral

$$\int_{\partial D} (x^2y - 2y)dx + (2x - y^2x)dy$$

takes on its maximal possible value and compute this value.

10. Find out if the cross section of the quadratic surface $2x^2 = 3y^2 + 5z^2 - 1$ by the plane $z = x - y$ is a hyperbola.

11. For a square-shaped thin board of mass density σ and area a^2, compute the moment of inertia about the axis perpendicular to the plane of the square and passing through one of its vertices.

12. Compute the area of the part inside the cylinder $x^2 + y^2 = 12a^2$ of the surface obtained by rotating the parabola $z = x^2/a$ lying in the plane $y = 0$ about the z-axis.

13. Give an example of a curl-free vector field which is not the gradient of any function.

The remaining three multipart problems form one of recent exams

14. Consider function $f = x^2 - 3xy + 3y^2$ in the region $D : x^2 - xy + y^2 \leq 3$.

(a) Prove that the boundary curve ∂D is an ellipse, and that the region D is bounded.

(b) Find principal axes of the ellipse ∂D and compute the area of D.

(c) Find critical points of f restricted to the boundary ∂D.

(d) Locate critical points of f inside D and determine if they are local minima, maxima, or saddles.

(e) Find the maximum and minimum values of f in D.

(f) Orient ∂D as the boundary of D, and compute the circulation of ∇f along ∂D and the flux of ∇f across ∂D.

(g) Denote by $D(t)$ the region into which D is carried by the flow of the vector field ∇f after time t. Compute the area $A(t)$ of $D(t)$.

15. The following questions are about differentiable vector fields in three dimensions.

(a) Give an example of a vector field which is curl-free (i.e. has zero curl) but is not conservative. (Don't forget to prove that it is not conservative.)

(b) Give an example of a vector field whose flux across the boundary of every region is equal to the volume of that region. Justify your answer.

(c) Give an example of a non-zero vector field which is curl-free and divergence-free (i.e. has zero curl and zero divergence). In the domain of this vector field:

- Does there exist a closed oriented curve C such that the circulation of the vector field along C is non-zero? Why?
- Does there exist a closed oriented surface S such that the flux of the vector field across this surface is non-zero? Why?

(d) Let $\vec{\omega}$ be a constant vector field. Give an example of a divergence-free vector field whose curl is $\vec{\omega}$.

- Does there exist a closed oriented curve C such that the circulation of the vector field along C is non-zero? Why?
- Does there exist a closed oriented surface S such that the flux of the vector field across this surface is non-zero? Why?

16. This problem is about areas of parametric surfaces in space.

(a) Give the definition of surface area of a parametric surface. (*Remark:* The answer should have the form of a double integral.)

(b) Derive an integral formula for the area of the surface of revolution described in cylindrical coordinates by equation $r = f(z)$, where f is a given differentiable function on the interval $a \le z \le b$. (*Remark:* The answer should have the form of a single-variable integral.)

(c) Compute the area of the surface described by the equation $z^6 = x^2+y^2$ and inequalities $0 \le z \le 1/2$. (*Remark:* The answer happens to have the form $\pi a/3^3 4^3$, where a is a 2-digit integer. State your answer by finding a.)