Spring 2015. Math 104. Sample problems

1. A (real or complex) number x is called *algebraic* if it is a root of a non-zero polynomial with integer coeficients, and *transcendental* otherwise. Prove that the set of algebraic numbers is countable, and derive that transcendental numbers exist.

2. Is there an uncountable closed subset of $\mathbf{R}$ which contains no rational numbers?

3. Prove that every open subset in $\mathbf{R}$ is the union of at most countably many disjoint open intervals (finite or infinite).

4. Prove that the function equal 0 for $x \leq 0$ and equal $e^{-1/x}$ for $x > 0$ is infinitely differentiable at $x = 0$ and has all Taylor coefficients equal to 0.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function such that $|f'(x)| < 0.99$ for all $x \in \mathbf{R}$. Prove that f has exactly one fixed point x_0 (i.e. solution to $f(x_0) = x_0$), and that every sequence (x_n) defined by an arbitrary choice of x_1 and by the recursion relation $x_{n+1} = f(x_n)$ for all $n = 1, 2, 3, \dots,$ converges to x_0.

6. Give an example of a differentiable function $f : \mathbf{R} \to \mathbf{R}$ without fixed points, and such that $|f'(x)| < 1$ for all $x \in \mathbf{R}$.

7. Prove that the sequence defined by the recursion relation $x_{n+1} = (x_n + a x_n^{-1})/2$, where a is a given positive number, and x_1 any positive number, converges to $\sqrt{a}$. (Try this algorithm of computing $\sqrt{a}$ on a calculator to see how quickly it converges. Can you explain the high convergence rate?)

8. Prove that sequence $s_n := 1 + 1/2 + 1/3 + \cdots + 1/n - \log n$ converges to a limit C between 0 and 1 (called *Euler's constant*).

9. Prove that a continuous function $f : \mathbf{R} \to \mathbf{R}$ mapping open sets into open sets is monotone.

10. Prove that every continuous mapping $f : [0, 1] \to [0, 1]$ has a fixed point (i.e. a solution to $f(x) = x$).

11. Prove that function $f : \mathbf{R} \to \mathbf{R}$ is continuous if and only its graph, $graph(f) := \{(x, y) \in \mathbf{R}^2 \mid y = f(x)\}$, is a closed subset in $\mathbf{R}^2$.

12. Suppose that a real function defined on $\mathbf{R}$ satisfies $\lim_{h\to 0} (f(x + h) - f(x - h)) = 0$ for every $x \in \mathbf{R}$. Does it imply that f is continuous?

13. Prove that a real-valued function defined on a dense subset $X \subset [0, 1]$ extends to a continuous function on $[0, 1]$ if and only if it is uniformly continuous on X.

14. Suppose $c_0 + c_1/2 + +c_2/3 + \cdots + c_n/(n + 1) = 0$, where $c_0, \ldots , c_n$ are real numbers. Prove that polynomial $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ has a real root between 0 and 1.

15. Let p and q be positive numbers such that $1/p+1/q = 1$. Prove *Young's inequality*: for all $u, v \geq 0$, we have $uv \leq u^p/p + v^q/q$, where the equality holds if and only if $u^p = v^q$.

16. Let f, g be nonnegative functions Riemann-integrable on $[a, b]$, and such that $\int_a^b f^p dx = 1 = \int_a^b g^q dx$ $(1/p+1/q = 1)$. Show, using the previous problem, that $\int_a^b fgdx \leq 1$, and derive *Hölder's inequality*: if $f, g \in \mathcal{R}[a, b]$, then

$$\left|\int_a^b fgdx\right| \leq \left(\int_a^b |f|^p dx\right)^{1/p} \left(\int_a^b |g|^q dx\right)^{1/q},$$

where $p, q > 0$ satisy $1/p + 1/q = 1$. (The special case $p = q = 2$ is called *Cauchy–Schwarz's inequality*).

17. Suppose a continuous function $f : [0, 1] \to \mathbf{R}$ satisfies $\int_0^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0$ on $[0, 1]$.

18. Let (f_n) be a uniformly bounded sequence of functions Riemann-integrable on $[a, b]$. Prove that the sequence $F_n(x) := \int_a^x f_n(t)dt$ of functions of $x \in [a, b]$ contains a uniformly convergent subsequence.

19. Prove that a nested sequenc of non-empty compact sets has non-empty intersection.

20. Prove that the intersection of countably many open dense subsets is dense. (This is called the *Baire category theorem.*)

21. Prove that $\lim_{n\to\infty}(1 + 1/n)^n$ exists and is equal to $e := \sum_{k=0}^{\infty} 1/k!$.

22. Prove that the sequence $P_N := \prod_{k=1}^{N} \frac{1}{1-1/p_k}$, where $p_1 = 2, p_2 = 3, \ldots, p_k, \ldots$ are consecutive prime numbers, tends to $+\infty$, and derive that the series $\sum 1/p_k$ diverges.