PERMUTATION-EQUIVARIANT QUANTUM K-THEORY V. TORIC q-HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We first retell in the K-theoretic context the heuristics of S^1-equivariant Floer theory on loop spaces which gives rise to D_q-module structures, and in the case of toric manifolds, vector bundles, or super-bundles to their explicit q-hypergeometric solutions. Then, using the fixed point localization technique developed in Parts II–VI, we prove that these q-hypergeometric solutions represent K-theoretic Gromov-Witten invariants.

S^1-equivariant Floer theory

We recall here our old (1994) heuristic construction [5, 6] which highlights the role of D-modules in quantum cohomology theory, and adjust the construction to the case of quantum K-theory and D_q-modules, following the more recent exposition [8].

Let X be a compact symplectic (or Kähler) target space, which for simplicity is assumed simply-connected, and such that $\pi_2(X) = H_2(X) \cong \mathbb{Z}^K$. Let $d = (d_1, \dots, d_k)$ be integer coordinates on $H_2(X)$, and $\omega_1, \dots, \omega_K$ be closed 2-forms on X with integer periods, representing the corresponding basis of $H^2(X, \mathbb{R})$.

On the space L_0X of contractible parameterized loops $S^1 \to X$, as well as on its universal cover $\widetilde{L_0X}$, one defines closed 2-forms Ω_i, which associates to two vector fields ξ and η along a given loop the value

$$\Omega_i(\xi, \eta) := \oint \omega_i(\xi(t), \eta(t))\ dt.$$

A point $\gamma \in \widetilde{L_0X}$ is a loop in X together with a homotopy type of a disk $u : D^2 \to X$ attached to it. One defines the *action functionals*

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$H_i : \widetilde{L_0X} \to \mathbb{R}$ by evaluating the 2-forms ω_i on such disks:

$$H_i(\gamma) := \int_{D^2} u^* \omega_i.$$

Consider the action of S^1 on $\widetilde{L_0X}$, defined by the rotation of loops, and let V denote the velocity vector field of this action. It is well-known that V is Ω_i-hamiltonian with the Hamilton function H_i, i.e.:

$$\mathrm{i}_V \, \Omega_i + dH_i = 0, \quad i = 1, \dots, K.$$

Denote by z the generator of the coefficient ring $H^*(BS^1)$ of S^1-equivariant cohomology theory. The S^1-equivariant de Rham complex (of $\widetilde{L_0X}$ in our case) consists of S^1-invariant differential forms with coefficients in $\mathbb{R}[z]$, and is equipped with the differential $D := d + zi_V$. Then the degree-2 elements

$$p_i := \Omega_i + zH_i, \quad i = 1, \dots, K,$$

are S^1-equivariantly closed: $Dp_a = 0$. This is standard in the context of Duistermaat–Heckman's formula.

Furthermore, the lattice $\pi_2(X)$ acts by deck transformations on the universal covering $\widetilde{L_0X} \to L_0X$. Namely, an element $d \in \pi_2(X)$ acts on $\gamma \in \widetilde{L_0X}$ by replacing the homotopy type $[u]$ of the disk with $[u]+d$. We denote by $Q^d = Q_1^{d_1} \cdots Q_K^{d_K}$ the operation of pulling-back differential forms by this deck transformation. It is an observation from [5, 6] that the operations Q_i and the operations of exterior multiplication by p_i do not commute:

$$p_i Q_{i'} - Q_{i'} p_i = -zQ_i \delta_{ii'}.$$

These are commutation relations between generators of the algebra of differential operators on the K-dimensional torus:

$$[-zQ_i \partial_{Q_i}, Q_{i'}] = -zQ_i \delta_{ii'}.$$

Likewise, if P_i denotes the S^1-equivariant line bundle on $\widetilde{L_0X}$ whose Chern character is e^{-p_i}, then tensoring vector bundles by P_i and pulling back vector bundles by Q_i do not commute:

$$P_i Q_{i'} = ((q-1)\delta_{ii'} + 1) Q_{i'} P_i.$$

These are commutation relations in the algebra of finite-difference operators, generated by multiplications and translations:

$$Q_i \mapsto Q_i \times, \quad P_i \mapsto e^{zQ_i \partial_{Q_i}} = q^{Q_i \partial_{Q_i}}, \quad \text{where} \quad q = e^z.$$

Thinking of these operations acting on S^1-equivariant Floer theory of the loop space, one arrives at the conclusion that S^1-equivariant Floer cohomology (K-theory) should carry the structure of a module over the algebra of differential (respectively finite-difference) operators. We will elucidate this conclusion with toric examples after giving a convenient description of toric manifolds.

Toric manifolds

Fans and momentum polyhedra are two the most popular languages in algebraic and symplectic geometry of toric manifolds [1]. In symplectic *topology*, a third framework, where toric manifolds are treated as symplectic reductions or GIT quotients of a linear space, turns out to be more convenient [4].

Let $\triangle$ be the momentum polyhedron of a compact symplectic toric manifold (we remind that it lives in the dual of the Lie algebra of a compact torus, and is therefore equipped with the integer lattice), and N be the number of its hyperplane faces. The corresponding N supporting affine linear functions with the minimal integer slopes canonically embed $\triangle$ into the first orthant $\mathbb{R}^N_+$ in $\mathbb{R}^N$, and thereby represent the toric manifold as the symplectic quotient of $\mathbb{C}^N$.

Indeed, the torus T^N acts by diagonal matrices on $\mathbb{C}^N$ with the momentum map $(z_1, \ldots, z_N) \mapsto (|z_1|^2, \ldots, |z_N|^2)$: $\mathbb{C}^N \to \mathbb{R}^N_+ \subset \mathrm{Lie}^*\, T^N$. For a subtorus $T^K \subset T^N$, the momentum map is obtained by further projection $\mathbf{m} : \mathrm{Lie}^*\, T^N \to \mathrm{Lie}^*\, T^K = \mathbb{R}^K$. The last equality uses a basis, $(p_1, \ldots, p_K)$, which we will assume integer. In fact one only needs to look at the *picture* $\mathbf{m}(\mathbb{R}^N_+) \subset \mathbb{R}^K$ of the first orthant (see example on Figure 1), i.e. to know the images $u_1, \ldots, u_N$ in $\mathbb{R}^K$ of the unit coordinate vectors from $\mathbb{R}^N$:

$$u_j = p_1 m_{1j} + \cdots + p_K m_{Kj}, \quad j = 1, \ldots, N.$$

When $\triangle$ is the fiber $\mathbf{m}^{-1}(\omega)$ in the first orthant over some regular value ω, the initial toric manifold is identified with the symplectic reduction $X = \mathbb{C}^N//_\omega T^K$. Alternatively, removing from $\mathbb{C}^N$ all coordinate subspaces whose moment images do not contain ω, one identifies X with the quotient $\mathring{\mathbb{C}}^N/T^K_{\mathbb{C}}$ of the rest by the action of the compexified torus (GIT quotient), and thereby equips X with a complex structure.

Here is how basic topological information about X can be read off the *picture*. The space $\mathbb{R}^K$ and the lattice spanned by p_i are identified with $H^2(X, \mathbb{R}) \supset H^2(X, \mathbb{Z})$. The vectors $u_1, \ldots, u_N$ represent cohomology classes of the toric divisors of complex codimension 1 (they correspond to the hyperplane faces of the momentum polyhedron), and $c_1(T_X)$ is their sum. In the example of Figure 1, $u_1 = u_2 = p_1, u_2 = p_2, u_4 = p_2 - p_1$. The *chamber* (connected component) of the set of regular values of the moment map, which contains ω (it is the darkest region

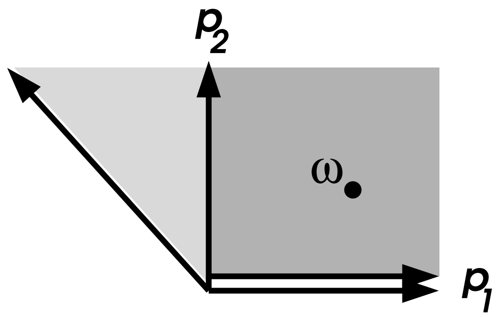

FIGURE 1. $X = \mathrm{proj}(\mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1})$

on Figure 1) becomes the Kähler cone of X. It is the intersection of the images of those K-dimensional walls $\mathbb{R}_+^K$ of the first orthant $\mathbb{R}_+^N$ which contain ω in their image. In the example, there are 4 of these: spanned by (u_1, u_3), (u_2, u_3), (u_1, u_4), and (u_2, u_4). They are in one-to-one correspondence with the vertices of the momentum polyhedron, and hence with fixed points of T^N in X. By the way, X is non-singular if and only if the determinants of these maps $\mathbb{R}_+^K \to \mathbb{R}^K$ (i.e. appropriate $K \times K$ minors of the $K \times N$ matrix $\mathbf{m}$) are equal to ± 1.

The ring $H^*(X)$ is multiplicatively generated by $u_1, \ldots, n_N$, which besides the $N - K$ linear relations (given by the above expressions in terms of $p_1, \ldots, p_K$,) satisfy multiplicative *Kirwan's relations*. Namely, $\prod_{j \in J} u_j = 0$ whenever the toric divisors u_j with $j \in J \subset \{1, \ldots, N\}$ have empty geometric intersection. In minimalist form, for each *maximal* subset $J \subset \{1, \ldots, N\}$ such that the cone spanned by the vectors u_j on the *picture* misses the Kähler cone, there is one Kirwan's relation $\prod_{j \notin J} u_j = 0$. In our example, there are two Kirwan's relations: $u_1 u_2 = 0$ and $u_3 u_4 = 0$, i.e. the complete presentation of $H^*(X)$ is $p_1^2 = p_2(p_2 - p_1) = 0$.

The spectrum of the algebra defined by Kirwan's relations (we call it *hedgehog*, after Czech, or anti-tank hedgehogs), is described geometrically as follows. For each $\alpha \in X^T$, consider the corresponding K-dimensional wall of $\mathbb{R}_+^N$ whose picture contains ω, and let $J(\alpha)$ denote the corresponding cardinality-K subset of $\{1, \ldots, N\}$. In the complex space with coordinates $u_1, \ldots, u_N$, consider the $N-K$-dimensional coordinate subspace (*rail*) $\mathbb{C}_\alpha^{N-K}$ given by the equations $u_j = 0$, $j \in J(\alpha)$. The hedgehog is the union of the rails. Respectively $H^*(X, \mathbb{C})$ is the algebra of functions on the "thick point", obtained by intersecting the hedgehog with the K-dimensional range of the map $\mathbb{C}^K \to \mathbb{C}^N : p \mapsto u = \mathbf{m}^t p$. In the T^N-equivariant version of the theory, this subspace is deformed into

$$u_j(p) = \sum_{i=1}^{K} p_i m_{ij} - \lambda_j, \quad j = 1, \dots, N,$$

and for generic λ intersects the hedgehog at isolated points corresponding to the fixed points αX^T. Here λ_j are the generators of the coefficient ring $H^*(BT^N)$. Finally, the operation of integration $H^*_{T^N}(X) \to H^*_{T^N}(pt)$ can be written (under some orientation convention) in the form of the residue sum over these intersection point:

$$\int_X \phi(p.\lambda) = \sum_{\alpha \in X^T} \mathrm{Res}_{p:u(p)\in \mathbb{C}^{N-K}_\alpha} \frac{\phi(p,\lambda)\, dp_1 \wedge \cdots \wedge dp_K}{u_1(p)\dots u_N(p)}.$$

This follows from fixed point localization.

In K-theory, let P_i and U_j be the (T^N-equivariant) line bundles whose Chern characters are e^{-u_j} and e^{-p_i} respectively. The ring $K^0_{T^N}(X)$ is described by Kirwan's relations

$$\prod_{j\in J}(1-U_j) = 0 \text{ whenever } \bigcap_{j\in J} u_j = \emptyset \text{ for corresponding toric divisors,}$$

together with the multiplicative relations

$$U_j = \prod_{i=1}^{k} P_i^{m_{ij}} \Lambda_j^{-1}, \text{ where } \Lambda_j = e^{-\lambda_j} \text{ are generators of } \mathrm{Repr}(T^N),$$

the coefficient ring of T^N-equivariant K-theory. The K-theoretic hedgehog, defined by Kirwan's relations, lives in the complex torus $(\mathbb{C}^\times)^N$ with coordinates $U_1, \dots, U_N$, and is the union of subtori $(\mathbb{C}^\times)^{N-K}_\alpha$ given by the equations $U_j = 1, j \in J(\alpha)$. The trace operation $\mathrm{tr}_{T^N} : K^0_{T^N}(X) \to K^0_{T^N}(pt)$ takes on the residue form

$$\mathrm{tr}_{T^N}(X; \Phi(P)) = \sum_{\alpha\in X^T} \mathrm{Res}_{P:U(P)\in(\mathbb{C}^\times)^{N-K}_\alpha} \frac{\Phi(P)\, dP_1 \wedge \cdots \wedge dP_K}{\prod_{j=1}^N (1-U_j(P))\, P_1 \cdots P_K}.$$

This follows from Lefschetz' fixed point formula. In the example, we have: $U_1 = P_1/\Lambda_1$, $U_2 = P_1/\Lambda_2$, $U_3 = P_2/\Lambda_3$, $U_4 = P_2/P_1\Lambda_4$. There are four intersections with the hedgehog: $U_1 = U_3 = 1$, $U_2 = U_3 = 1$, $U_1 = U_4 = 1$, and $U_2 = U_4 = 1$. The respective residues take the form:

$$\frac{\Phi(\Lambda_1,\Lambda_3)}{(1-\Lambda_1/\Lambda_2)(1-\Lambda_3/\Lambda_1\Lambda_4)} + \frac{\Phi(\Lambda_2,\Lambda_3)}{(1-\Lambda_2/\Lambda_1)(1-\Lambda_3/\Lambda_2\Lambda_4)} +$$
$$\frac{\Phi(\Lambda_1,\Lambda_1\Lambda_4)}{(1-\Lambda_1/\Lambda_2)(1-\Lambda_1\Lambda_4/\Lambda_3)} + \frac{\Phi(\Lambda_2,\Lambda_2\Lambda 4)}{(1-\Lambda_2/\Lambda_1)(1-\Lambda_2\Lambda_4/\Lambda_3)}.$$

Linear sigma-models

Returning to the heuristics based on loop spaces, we replace the universal cover $\widetilde{L_0 X}$ of the space of contractible loops in $X = \mathbb{C}^N//_\omega T^K$ with the infinite dimensional toric manifold $L\mathbb{C}^N//_\omega T^K$. Note that the group LT^K of *loops* in T^K is homotopically the same as $T^K \times \pi_1(T^K)$, and that neglecting to factorize by $\pi_1(T^K) = \mathbb{Z}^K$ is equivalent to passing to the universal cover of $L_0 X$. We consider the model of the loop space $L\mathbb{C}^N = \mathbb{C}^N[\zeta, \zeta^{-1}]$ as equivariant with respect to $T^N \times S^1$, where T^N acts as before on $\mathbb{C}^N$, and S^1 acts by rotation of the loop's parameter: $\zeta \mapsto e^{it}\zeta$. The *picture* in $\mathbb{R}^K$ corresponding to our infinite dimensional toric manifold consists of countably many copies of each of the vectors u_j. The copies represent the Fourier modes of the loops, and the equivariant classes of the corresponding toric divisors in terms of the basis $p_1, \ldots, p_K$ have the form

$$\sum_{i=1}^K p_i m_{ij} - \lambda_j - rz = u_j(p) - rz, \quad j = 1, \ldots, N, \; r = 0, \pm 1, \pm 2, \ldots$$

In K-theory of the loop space, the line corresponding line bundles are

$$\prod_{i=1}^K P_i^{m_{ij}} \Lambda_j^{-1} q^r = U_j(P) q^r, \quad j = 1, \ldots, N, \; r = 0, \pm 1, \pm 2, \ldots$$

The *Floer fundamental cycle* Fl_X in the loop space, by definition, consists of those loops which bound holomorphic disks. In our model of the loop space, $Fl_X = \mathbb{C}^N[\zeta]//_\omega T^K$. This gives rise to the following formula for the trace over Fl_X:

$$\operatorname{tr}_{T^N \times S^1}(Fl_X; \Phi(P)) =$$
$$\frac{1}{(2\pi i)^K} \oint \Phi(P) \, \frac{\prod_{j=1}^N \prod_{r=1}^\infty (1 - U_j(p) q^r)}{\prod_{j=1}^N \prod_{r=-\infty}^\infty (1 - U_j(P) q^r)} \frac{dP_1 \wedge \cdots \wedge dP_K}{P_1 \cdots P_K}.$$

Thus, the structure sheaf of the semi-infinite cycle Fl_X is Poincaré-dual to the semi-infinite product

$$\widehat{I}_X := \prod_{j=1}^N \prod_{r=1}^\infty (1 - U_j(P) q^r).$$

Similarly, for a toric bundle $E \to X$ or super-bundle ΠE (see Part IV), endowed with the fiberwise scalar action of T^1, $\widehat{I}_E$ and $\widehat{I}_{\Pi E}$ are obtained from $\widehat{I}_X$ by respectively division and multiplication by the K-theoretic

Euler class of the obvious semi-infinite vector bundle:

$$\widehat{I}_E := \widehat{I}_X / \prod_{a=1}^{L}\prod_{r=0}^{\infty}(1-\lambda V_a(P)q^{-r}), \text{ and } \widehat{I}_{\Pi E} = \widehat{I}_X \prod_{a=1}^{L}\prod_{r=0}^{\infty}(1-\lambda V_a(P)q^{-r}).$$

Here $\lambda \in T^1$, and V_a are toric line bundles,

$$V_a(P) = P_1^{l_{1a}} \cdots P_K^{l_{Ka}}, \quad E = \oplus_{a=1}^{L} V_a.$$

Our aim is to compute the left D_q-module generated by $\widehat{I}_X$. The deck transformation Q^d corresponding to a homology class $d \in H_2(X)$ acts in our model by $Q^d(P_i) = P_i q^{-d_i}$, where $(d_1, \ldots, d_K)$ are coordinates on $H_2(X)$ in the basis dual to $(p_1, \ldots, p_K)$. We find that $\widehat{I}_X$ satisfies the following relations:

$$Q^d \widehat{I}_X = \prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{D_j(d)-1}(1-U_j(P)q^{-r})}{\prod_{r=-\infty}^{-1}(1-U_j(P)q^{-r})}\, \widehat{I}_X^K.$$

Of course, the relations for all Q^d follow from the basis relations with $Q^d = Q_i, i = 1, \ldots, K$. For instance, in our example, after some rearrangements, we obtain a system of two finite-difference equations (for $d = (1,0)$ and $(0,1)$):

$$(1 - P_1\Lambda_1^{-1})(1 - P_2\Lambda_2^{-1})\widehat{I}_X = Q_1(1 - P_2P_1^{-1}\Lambda_4^{-1})\widehat{I}_X$$
$$(1 - P_2\Lambda_2^{-1})(1 - P_2P_1^{-1}\Lambda_4^{-1})\widehat{I}_X = Q_2\widehat{I}_X.$$

To save space, we refer the reader to [6] for an explanation (though given in the cohomological context) of how to mechanically pass from this "momentum" representation of the Floer fundamental class (i.e. expresses as a function of P) to the "coordinate" representation in the form of the hypergeometric Q-series with vector coefficients in $K^0(X)$. In that representation, Q^d acts as multiplication by $Q_1^{d_1} \cdots Q_K^{d_K}$, and P_i acts as $P_i q^{Q_i \partial_{Q_i}}$, i.e. as the change $Q_i \mapsto qQ_i$ accompanied with multiplication by P_i in $K^0(X)$. With these conventions, we have:

$$I_X = \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{0}(1-U_j(P)q^r)}{\prod_{r=-\infty}^{D_j(d)}(1-U_j(P)q^r)}.$$

This is just another way to describe the same D_q-module, and so I_X satisfies the system of finite-difference equations:

$$\prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{m_{ij}-1}(1-q^{-r}U_j(Pq^{Q\partial_Q}))}{\prod_{r=-\infty}^{-1}(1-q^{-r}U_j(Pq^{Q\partial_Q}))}\, I_X = Q_i I_X, \quad i = 1, \ldots, K.$$

Real life

The toric q-hypergeometric function I_X, though comes from heuristic manipulation, has something to do with real life.

`Theorem.` *The series $(1-q)I_X$ is a value of the big J-function in symmetrized T^N-equivariant quantum K-theory of toric manifold X.*

`Proof.` We follow the plan based on fixed point localization and explained in detail in Part II and Part IV in the example of complex projective spaces.

We write $I_X = \sum_{\alpha \in X^{T^N}} I_X^{(\alpha)} \phi_\alpha$ is components in the basis $\{\phi_\alpha\}$ of delta-functions of fixed points. Denote by $U_j(\alpha)$ the restriction of $U_j(P)$ to the fixed point α. We have $U_j(P) = 1$ for each of the K values of $j \in J(\alpha)$, i.e.

$$P_1^{m_{1j}} \cdots P_K^{m_{Kj}} = \Lambda_j, \ j \in J(\alpha).$$

This determines expressions for P_i, and consequently for $U_j(P)$ with $j \notin J(\alpha)$ as Laurent monomials in $\Lambda_1, \dots, \Lambda_N$. We have

$$I_X^{(\alpha)}(q) = \sum_{d \in \mathbb{Z}_+^K(\alpha)} \frac{Q^d}{\prod_{j \in J(\alpha)} \prod_{r=1}^{D_j(d)} (1-q^r)} \prod_{j \notin J(\alpha)} \frac{\prod_{r=-\infty}^{0} (1 - q^r U_j(\alpha))}{\prod_{r=-\infty}^{D_j(d)} (1 - q^r U_j(\alpha))}.$$

The summation range $\mathbb{Z}_+^K(\alpha)$ is over $d \in \mathbb{Z}^K$ such that $D_j(d) \geq 0$ for all $j \in J(\alpha)$, because outside this range, there is a factor $(1-q^0)$ in the numerator.[1]

(i) Temporarily encode degrees d by $D_j(d), j \in J(\alpha)$, i.e. introduce Laurent monomials $Q_j(\alpha)$ in Novikov's variables $Q_1, \dots, Q_K$ such that

$$Q_1^{d_1} \cdots Q_K^{d_K} = \prod_{j \in J(\alpha)} Q_j(\alpha)^{D_j(d)} \ \text{ for all } d.$$

We have:

$$\sum_{d \in \mathbb{Z}_+^K(\alpha)} \frac{Q^d}{\prod_{j \in J(\alpha)} \prod_{r=1}^{D_j(d)} (1-q^r)} = e^{\sum_{k>0} \sum_{j \in J(\alpha)} Q_j(\alpha)^k / k(1-q^k)}.$$

According to Part I (or Part III), the right hand side is $J_{pt}(\tau)/(1-q)$ with $\tau = \sum_{j \notin J(\alpha)} Q_j(\alpha)$ in the Novikov ring considered as a λ-algebra with the Adams operations $\Psi^k(Q^d) = Q^{kd}$. It follows now from results of Part IV, that $(1-q)I_X^{(\alpha)}$, expanded near the roots of unity (i.e. with the products on the right expanded as power series in q), represents

[1] It is dual in $\mathbb{Z}^K = H_2(X, \mathbb{Z})$ to the image $\mathbb{R}_+^K(\alpha)$ on the *picture* $\mathbb{R}^K = H^2(X, \mathbb{R})$ of the K-dimensional face of $\mathbb{R}_+^N$ corresponding to α. Note that the intersection of all $\mathbb{R}_+^K(\alpha)$ is the closure of the Kähler cone, and respectively the convex hull of all $\mathbb{Z}_+^K(\alpha)$ is the *Mori cone* of possible degrees of holomorphic curves.

a value of the big J-function $\mathcal{J}_{pt}$. Namely, recall fromPart IV that by Γ-operators we mean q-difference operators with symbold defined by

$$\Gamma_q(x) := e^{\sum_{k>0} x^k/k(1-q^k)} \sim \prod_{r=0}^{\infty} \frac{1}{1-xq^r}$$

Expressing each u_j as a linear combination $u_j = \sum_{i\in J(\alpha)} u_i n_{ij}$, we find for multi-variable Γ-operators:

$$\frac{\Gamma_{q^{-1}}(\lambda)}{\Gamma_{q^{-1}}\left(\lambda\, q^{\sum_{i\in J(\alpha)} n_{ij} Q_i(\alpha)\partial_{Q_i(\alpha)}}\right)} Q^d = Q^d \frac{\prod_{r=-\infty}^{0}(1-\lambda q^r)}{\prod_{r=-\infty}^{D_j(d)}(1-\lambda q^r)}.$$

Thus, applying to $J_{pt}/(1-q)$ such operators (one for each $j=1,\dots,N$) and setting $\lambda = U_j(\alpha)$, we obtain $I_X^{(\alpha)}$. Due to the invariance of the big J-function $\mathcal{J}_{pt}$ with respect to the q-difference operators (as explained in Part IV), we conclude that $(1-q)I_X^{(a)}$ is a value of $\mathcal{J}_{pt}$.

(ii) All poles of $I_X^{(\alpha)}$ away from roots of unity are simple for generic values of $\Lambda_1,\dots,\Lambda_N$. We compute the residues at such poles. The pole is specified by the choice in the denominators of one of the factors $1-q^m U_{j_0}$ with a $j_0 \notin J(\alpha)$, and by the choice of one of the mth roots $\lambda^{1/m}$ of $\lambda := U_{j_0}(\alpha)$. The choice of $j_0 \notin J(\alpha)$ determines a 1-dimensional orbit of $T^N_{\mathbb{C}}$ in X, connecting the fixed point α with another fixed point, β. The closure of this orbit is a holomorphic sphere $\mathbb{C}P^1 \subset X$, represented on the *picture* by a collection of u_j of cardinality $K+1$: the union $J(\alpha)\sqcup\{j_0\} = J(\beta)\sqcup\{j_0'\}$, where j_0' is a unique element of $J(\alpha)$ missing in $J(\beta)$. The torus T^N acts on the cotangent lines to this $\mathbb{C}P^1$ at the fixed points by the characters $\lambda = U_{j_0}(\alpha)$ and $\lambda^{-1} = U_{j_0'}(\beta)$ respectively (which are therefore inverse to each other). Moreover, denote by $d_{\alpha\beta}$ the degree of this $\mathbb{C}P^1$. Then $U_j(\alpha)/U_j(\beta) = \lambda^{D_j(d_{\alpha\beta})}$, and in particular $D_{j_0}(d_{\alpha\beta}) = D_{j_0'}(d_{\alpha\beta}) = 1$. Indeed, by cohomological fixed point localization on this $\mathbb{C}P^1$,

$$D_j(d_{\alpha\beta}) := -\int_{d_{\alpha\beta}} \ln U_j = \frac{-\ln U_j(\alpha)}{-\ln\lambda} + \frac{-\ln U_j(\beta)}{\ln\lambda} = \frac{\ln U_j(\alpha)/U_j(\beta)}{\ln\lambda}.$$

Consequently, at $q=\lambda^{-1/m}$ we have for all r and j:

$$1-q^r U_j(\alpha) = 1-q^{r-mD_j(d_{\alpha\beta})}U_j(\beta).$$

Under these constraints,

$$\frac{\prod_{r=-\infty}^{mD_j(d_{\alpha\beta})}(1-q^rU_j(\alpha))}{\prod_{r=-\infty}^{D_j(d)}(1-q^rU_j(\alpha))} = \frac{\prod_{r=-\infty}^{0}(1-q^rU_j(\beta))}{\prod_{r=-\infty}^{D_j(d)-mD_j(d_{\alpha\beta})}(1-q^rU_j(\beta))}.$$

It follows that at $q=\lambda^{-1/m}$,

$$(1-q^m\lambda)I_X^{(\alpha)}(q)=(1-q^mU_{j_0}(\alpha))\sum_{d\in\mathbb{Z}^K}Q^d\prod_{j=1}^N\frac{\prod_{r=-\infty}^0(1-q^rU_j(\alpha))}{\prod_{r=-\infty}^{D_j(d)}(1-q^rU_j(\alpha))}=$$

$$Q^{md_{\alpha\beta}}\,(1-q^mU_{j_0}(\alpha))\prod_{j=1}^N\frac{\prod_{r=-\infty}^0(1-q^rU_j(\alpha))}{\prod_{r=-\infty}^{mD_j(d_{\alpha\beta})}(1-q^rU_j(\alpha))}\times$$

$$\sum_{d\in\mathbb{Z}^K}Q^{d-md_{\alpha\beta}}\prod_{j=1}^N\frac{\prod_{r=-\infty}^0(1-q^rU_j(\beta))}{\prod_{r=-\infty}^{D_j(d)-mD_j(d_{\alpha\beta})}(1-q^rU_j(\beta))}.$$

Equivalently,

$$\mathrm{Res}_{q=\lambda^{-1/m}}\,I_X^{(\alpha)}(q)\,\frac{dq}{q}=-\frac{Q^{md_{\alpha\beta}}}{m}\,\frac{\phi^\alpha}{C_{\alpha\beta}(m)}\,I_X^{(\beta)}(\lambda^{-1/m}),$$

where $\phi^\alpha=\prod_{j\notin J(\alpha)}(1-U_j(\alpha))=\mathrm{Euler}_{T^N}^K(T_\alpha^*X)$, and

$$\frac{C_{\alpha\beta}(m)}{\phi^\alpha}=\phi^\alpha\prod_{r=1}^{m-1}(1-\lambda^{r/m})\prod_{j\neq j_0}\frac{\prod_{r=-\infty}^{mD_j(d_{\alpha\beta})}(1-\lambda^{-r/m}U_j(\alpha))}{\prod_{r=-\infty}^0(1-\lambda^{-r/m}U_j(\alpha))}.$$

Thus, the residues at the simple poles satisfy the recursion relations derived by fixed point localization arguments in Part II. More precisely, it remains to check that $C_{\alpha\beta}(m)=\mathrm{Euler}_{T^N}^K(T*_p\,\overline{\mathcal{M}}_{0,2}(X,md_{\alpha,b})$, where $T*_p$ is the virtual cotangent space to the moduli space at the point p represented by the n-multiple cover of the 1-dimensional orbit connecting fixed points α and β. This verification is straightforward. In $K^0_{T^N}(X)$, we have $T^*X=U_1+\dots+U_N-K$ (as follows from the quotient description of X, or by localization to X^T). Therefore the cotangent $T*_p$ is identified with $\oplus_jH^0(\mathbb{C}P^1;U_j^m)\ominus H^1(\mathbb{C}P^1;U_j^N-K-1$ (the last -1 stands for reparameterizations of $\mathbb{C}P^1$ with 2 marked points), which is easily described in terms of spaces of binary forms of degrees $D_j(\alpha\beta)$. The factors in the formula for $C_{\alpha\beta}(m)$ correspond to T^N-weight of the monomials in such binary forms.

From (i) and (ii) it follows that $(1-q)I_X$ is a value of the big J-function in permutation-equivariant quantum K-theory of X. Since I_X is defined over the λ-algebra $\mathbb{Z}[\Lambda^{\pm1}][[Q]]$ involving only Novikov's variables and functions on T^N, the value actually belongs to the *symmetrized* theory, i.e. carries information only about multiplicities the part of sheaf cohomology, *invariant* under permutations of marked points. □

In the case of bundle E or super-bundle ΠE, where $E = \oplus_{a=1}^{L} V_a$ is the sum of toric line bundles $V_a = \prod_i P_i^{l_{ia}}$, one similarly obtains q-hypergeometric series

$$I_E = \sum_{d\in\mathbb{Z}^K} Q^d \prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{0}(1-q^r U_j(P))}{\prod_{r=-\infty}^{D_j(d)}(1-q^r U_j(P))} \prod_{a=1}^{L} \frac{\prod_{r=-\infty}^{0}(1-\lambda q^r V_a(P))}{\prod_{r=-\infty}^{\triangle_a(d)}(1-\lambda q^r V_a(P))},$$

$$I_{\Pi E} = \sum_{d\in\mathbb{Z}^K} Q^d \prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{0}(1-q^r U_j(P))}{\prod_{r=-\infty}^{D_j(d)}(1-q^r U_j(P))} \prod_{a=1}^{L} \frac{\prod_{r=-\infty}^{\triangle_a(d)}(1-\lambda q^r V_a(P))}{\prod_{r=-\infty}^{0}(1-\lambda q^r V_a(P))},$$

where $\triangle_a(d) = \sum_i d_i l_{ia}$.

`Theorem.` *Functions $(1-q)I_E$ and $(1-q)I_{\Pi E}$ represent some values of the big J-functions in symmetrized quantum K-theories of toric bundle space E and super-bundle ΠE respectively.*

This theorem is proved the same way as the previous one.

The above results are K-theoretic analogues of cohomological "mirror formulas" [7, 3, 9]. The strategy we followed is due to J. Brown [2]. Some special cases were obtained in [8] by a different strategy.

References

[1] M. Audin. *The topology of torus actions on symplectic manifolds.* Birkhäuser, Basel, 1991.

[2] J. Brown. *Gromov–Witten invariants of toric fibrations.* IMRN, 2013, 46 p., arXiv:0901.1290

[3] T. Coates, A. Givental. *Quantum Riemann–Roch, Lefschetz and Serre.* Ann. of Math. (2), 165 (2007), 15–53.

[4] A. Givental. *A symplectic fixed point theorem for toric manifolds.* Preprint UC Berkeley, 1992. Published in: The Floer memorial volume. H. Hofer, C. H. Taubes, A. Weinstein, E. Zehnder (eds.), Progress in Math. 133, Birkhäuser, 1995, 445–481.

[5] A. Givental. *Homological geometry and mirror symmetry.* Talk at ICM-94. In: Proceedings of ICM-94 Zurich. Birkhäuser, Basel, 1995, 472–480.

[6] A. Givental. *Homological geometry I. Projective hypersurfaces.* Selecta Math. (New Series) 1 (1995), 325–345.

[7] A. Givental. *A mirror theorem for toric complete intersections.* Topological field theory, primitive forms and related topics (Kyoto, 1996), 1998, pp.141–175.

[8] A. Givental, V. Tonita. *The Hirzebruch-Riemann–Roch theorem in true genus-0 quantum K-theory.* Preprint, arXiv:1106.3136

[9] H. Iritani. *Convergence of quantum cohomology by quantum Lefschetz.* J. Reine Angew. Math. 610 (2007), 29–69.