# PERMUTATION-EQUIVARIANT QUANTUM K-THEORY VIII. EXPLICIT RECONSTRUCTION

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Abstract. In Part VII, we proved that the range $\mathcal{L}_X$ of the big J-function in permutation-equivariant genus-0 quantum K-theory is an overruled cone, and gave its adelic characterization. Here we show that the ruling spaces are $D_q$-modules in Novikov's variables, and moreover, that the whole cone $\mathcal{L}_X$ is invariant under a large group of symmetries of $\mathcal{L}_X$ defined in terms of $q$-difference operators. We employ this for the explicit reconstruction of $\mathcal{L}_X$ from one point on it, and apply the result to toric $X$, when such a point is given by the $q$-hypergeometric function.

## Adelic characterization

We begin where we left in Part VII: at a description of the range $\mathcal{L} \subset \mathcal{K}$ in the space $\mathcal{K}$ of $K^0(X) \otimes \Lambda$-value rational functions of $q$ of the J-function of permutation-equivariant quantum K-theory of a given Kähler target space $X$:

$$\mathcal{J} := 1 - q + \mathbf{t}(q) + \sum_\alpha \phi_\alpha \sum_{n,d} Q^d \langle \frac{\phi^\alpha}{1-qL}; \mathbf{t}(L), \dots, \mathbf{t}(L) \rangle_{0,1+n,d}^{S_n}.$$

We proved that $\mathcal{L}$ is an overruled cone, i.e. it is swept by a family of certain $\Lambda[q, q^{-1}]$-modules, called *ruling spaces*:

$$\mathcal{L} = \bigcup_{t \in \Lambda_+} (1-q)\, S(q)_t^{-1}\, \mathcal{K}_+,$$

where $S_t$ is a certain family of "matrix" functions rational in $q$, whose construction we are not going to remind here. Let us recall the *adelic characterization* of $\mathcal{L}$, which will be our main technical tool.

It is given in terms of the overruled cone $\mathcal{L}^{fake} \subset \widehat{K}$ in the space of vector-valued Laurent series in $q-1$, describing the range of the

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J-function in *fake* quantum K-theory of $X$:

$$\mathcal{L}^{fake} = \bigcup_{t\in\Lambda}(1-q)T_t, \quad T_t = S^{fake}(q)_t^{-1}\widehat{\mathcal{K}}_+.$$

Here $S_t^{fake}$ is some "matrix" Laurent series, $T_t$ is a tangent space to $\mathcal{L}^{fake}$, containing $(1-q)T_t$, and tangent to $\mathcal{L}^{fake}$ at all points of the ruling space $(1-q)T_t$.

According to the last section of Part VII, *a rational function* $\mathbf{f}\in\mathcal{K}$ *lies in* $\mathcal{L}$ *if and only if its Laurent series expansions* $\mathbf{f}_{(\zeta)}$ *near* $q=1/\zeta$ *satisfy the following three conditions:*

*(i)* $\mathbf{f}_{(1)}\in\mathcal{L}^{fake}$*;*

*(ii) when* $\zeta\neq 0,1,\infty$ *is a primitive* $m$*th root of unity,*

$$\mathbf{f}_{(\zeta)}(q^{1/m}/\zeta)\in\mathcal{L}_t^{(\zeta)},$$

*a certain subspace in* $\widehat{K}$*, determined by the tangent space* $T_t$ *to* $\mathcal{L}^{fake}$ *at the point* $\mathbf{f}_{(1)}$*;*

*(iii) when* $\zeta\neq 0,\infty$ *is not a root of unity,* $\mathbf{f}_{(\zeta)}$ *is a power series in* $q-1/\zeta$*, i.e.* $\mathbf{f}$ *has no pole at* $q=1/\zeta$*.*

The subspace $\mathcal{L}_t^{(\zeta)}$ is described as $\nabla_\zeta\Psi^m(T_t)\otimes_{\Psi^m(\Lambda)}\Lambda$, where the Adams operation $\Psi^m$ acts by $\Psi^m(q)=q^m$ and naturally on the $\lambda$-algebra $K^0(X)\otimes\Lambda$, and $\nabla_\zeta$ is the operator of multiplication by

$$e^{\sum_{k>0}\left(\frac{\Psi^k(T_X^*)}{k(1-\zeta^{-k}q^{k/m})}-\frac{\Psi^{km}(T_X^*)}{k(1-q^{km})}\right)}.$$

In its turn, the cone $\mathcal{L}^{fake}\subset\widehat{K}$ (and hence its tangent spaces $T_t$) can be expressed in terms of the cone $\mathcal{L}^H\subset\mathcal{H}$, describing the range of cohomological J-function in the space $\mathcal{H}$ of Laurent series in one indeterminate $z$ with coefficients in $H^{even}(X)\otimes\Lambda$. Namely, according to the Hirzebruch–Riemann–Roch theorem [2] in fake quantum K-theory,

$$\operatorname{qch}(\mathcal{L}^{fake})=\triangle\,\mathcal{L}^H,$$

where the *quantum Chern character* $\operatorname{qch}:\widehat{K}\to\mathcal{H}$ acts by $\operatorname{qch}q=e^z$, and by the natural Chern character $\operatorname{ch}:K^0(X)\otimes\Lambda\to H^{even}(X)\otimes\Lambda$ on the vector coefficients, while $\triangle$ acts as the multiplication in the classical cohomology of $X$ by the *Euler–Maclaurin asymptotics* (see [3, 2, 6]) of the infinite product:

$$\triangle\sim\prod_{r=1}^{\infty}\operatorname{td}(T_X\otimes q^{-r}).$$

Using all these descriptions, we are going on explore how the *string and divisor equations* of quantum cohomology theory manifest in the genus-0 permutation-equivariant quantum K-theory.

## Divisor equations and $\mathcal{D}_q$-modules

Let $p_1, \dots, p_K$ be a basis in $H^2(X, \mathbb{R})$ consisting if integer numerically effective classes, and let $Q^d = Q_1^{d_1} \cdots Q_K^{d_K}$, where $d_i = p_i(d)$, represent degree-$d$ holomorphic curves in the Novikov ring. We remind that the Novikov variables are included into the ground $\lambda$-algebra $\Lambda$.

The loop space $\mathcal{H}$ of Laurent series in $z$ with vector coefficients in $H^{even}(X) \otimes \Lambda$ is equipped with the structure of a module over the algebra $\mathcal{D}$ of differential operators in the Novikov variables, so that $Q_i$ acts as multiplication by $Q_i$, and $Q_i\partial_{Q_i}$ acts as $zQ_i\partial_{Q_i} - p_i$. The divisor equations in quantum cohomology theory imply (see e.g. [4]), that

*linear vector fields* $\mathbf{f} \mapsto (Q_i\partial_{Q_i} - p_i/z)\mathbf{f}$ *in* $\mathcal{H}$ *are tangent to* $\mathcal{L}^H \subset \mathcal{H}$.

In follows that *the ruling spaces (as well as tangent spaces) of* $\mathcal{L}^H$ *are* $\mathcal{D}$*-modules, i.e. are invariant with respect to each differential operator* $D(Q, zQ\partial_Q - p, z)$*, and moreover the flow* $\epsilon \mapsto e^{\epsilon D/z}$ *of the vector field* $\mathbf{f} \mapsto D\mathbf{f}/z$ *preserves* $\mathcal{L}^H$.

Indeed, for $\mathbf{f} \in \mathcal{L}^H$, the vector $(Q_i\partial_{Q_i} - p_i)\mathbf{f}$ lies in $T_{|f}\mathcal{L}^H$, and hence $(zO_i\partial_{Q_i} - p_i)\mathbf{f}$ lies in the same ruling space $zT_{\mathbf{f}}\mathcal{L}^H$ as $\mathbf{f}$ does. Therefore so does $D\mathbf{f}$, and hence $D\mathbf{f}/z \in T_{\mathbf{f}}\mathcal{L}^H$, i.e. the vector field $\mathbf{f} \mapsto D\mathbf{f}/z$ is tangent to $\mathcal{L}^H$.

Note that the operator $\triangle$ relating $\mathcal{L}^H$ and $\mathcal{L}^{\text{fake}}$ involves multiplication in the *commutative* classical cohomology algebra $H^{even}(X)$, but does *not* involve Novikov's variables. Consequently, *the tangent and ruling spaces of* $\mathrm{qch}(\mathcal{L}^{fake})$ *are* $D$*-modules* too, and moreover, *the flows* $\epsilon \mapsto e^{\epsilon D/z}$ *preserve* $\mathrm{qch}(\mathcal{L}^{fake})$.

We equip the space $\mathcal{K}$ of vector-valued rational functions of $q$ with the structure of a module over the algebra $\mathcal{D}_q$ of finite difference operators. It is generated (over the algebra of Laurent polynomials in $q$) by multiplication operators, acting as multiplications by $Q_i$, and translation operators, acting as $P_i q^{Q_i\partial_{Q_i}}$, where $P_i$ is the multiplication in $K^0(X)$ by the line bundle with the Chern character $\mathrm{ch}\, P_i = e^{-p_i}$.

`Proposition` (cf. [6, 4]). *The ruling spaces of the overruled cone* $\mathcal{L} \subset \mathcal{K}$ *of permutation-equivariant quantum K-theory is are* $\mathcal{D}_q$*-modules.*

`Proof.` If $\mathbf{f} \in \mathcal{L}$, it passes the tests (i),(ii),(iii) of adelic characterization. We need to show that $\mathbf{g} := P_i q^{Q_i\partial_{Q_i}}\mathbf{f}$, which obviously lies in $\mathcal{K}$, also passes the tests (and with the same $t \in K^0(X) \otimes \Lambda_+$). This is obvious for test (iii), and is true about test (i) because of the above $\mathcal{D}$-module (and hence $\mathcal{D}_q$-module) property of the ruling spaces $(1-q)T_t$ of $\mathcal{L}^{fake}$. To verify test (ii), we write:

$$\mathbf{g}_{(\zeta)}(q^{1/m}/\zeta) = P_i (q^{1/m})^{Q_i\partial_{Q_i}} \zeta^{-Q_i\partial_{Q_i}} \mathbf{f}_{(\zeta)}(q^{1/m}/\zeta).$$

First, note that the operator $\nabla_\zeta$ relating $\mathcal{L}^{(\zeta)}$ with $\Psi^m(T_t)$ does not involve Novikov's variables and commutes with $\mathcal{D}_q$. Next, let us elucidate the notation $\Psi^m(T_t)\otimes_{\Psi^m(\Lambda)}\Lambda$. In fact the space so indicated consists of linear combinations $\sum_a \lambda_a(Q,q)\Psi^m(\mathbf{f}_a)$, where $\mathbf{f}_a\in T_t$, and $\lambda_a\in\Lambda[[q-1]]$. We have the following commutation relations:

$$P_i\Psi^m=\Psi^m P_i^{1/m},\ \ (q^{1/m})^{Q_i\partial_{Q_i}}\Psi^m=q^{Q_i^m\partial_{Q_i^m}}\Psi^m=\Psi^m(q^{1/m})^{Q_i\partial_{Q_i}},$$
$$\zeta^{Q_i\partial_{Q_i}}\Psi^m=\zeta^{mQ_i^m\partial_{Q_i^m}}\Psi^m=\Psi^m.$$

Therefore

$$P_i(q^{1/m})^{Q_i\partial_{Q_i}}\zeta^{-Q_i\partial_{Q_i}}\left(\sum_a\lambda_a\Psi^m(\mathbf{f}_a)\right)=$$
$$\sum_a\left((q^{1/m}/\zeta)^{Q_i\partial_{Q_i}Q_i}\lambda_a\right)\ \Psi^m\left(P^{1/m}(q^{1/m})^{Q_i\partial_{Q_i}}\mathbf{f}_a\right),$$

which lies in $\Psi^m(T_t)\otimes_{\Psi^m(\Lambda)}\Lambda$ since $T_t$ is invariant under the operator $P^{1/m}(q^{1/m})^{Q_i\partial_{Q_i}}=e^{(zQ_i\partial_{Q_i}-p_i)/m}$. □

Let $D(Pq^{Q\partial_Q},q)$ be a constant coefficient finite difference operator, by which we mean a Laurent polynomial expression in translation operators $P_iq^{Q_i\partial_{Q_i}}$, and maybe $q$, with coefficients from $\Lambda$ *independent of* $Q$. We assume below that $\epsilon\in\Lambda_+$ to assure $\epsilon$-adic convergence of infinite sums.

`Theorem 1.` *The operator*

$$e^{\sum_{k>0}\Psi^k(\epsilon D(Pq^{kQ\partial_Q},q))/k(1-q^k)}$$

*preserves* $\mathcal{L}\subset\mathcal{K}$.

`Proof.` We show that if $(1-q)\mathbf{f}$ passes tests (i), (ii), (iii) of the adelic characterization of $\mathcal{L}$, then $(1-q)\mathbf{g}$, where

$$\mathbf{g}:=e^{\sum_{k>0}\Psi^k(\epsilon D(Pq^{kQ\partial_Q},q))/k(1-q^k)}\,\mathbf{f},$$

also does.

(i) Suppose $(1-q)\mathbf{f}_{(1)}$ lies in the ruling space $(1-q)T_t\subset\mathcal{L}^{fake}$. Note that the exponent $\sum_{k>0}\Psi^k(\epsilon D(Pq^{kQ\partial_Q},q))/k(1-q^k)$ has first order pole at $q=1$. According to the discussion above the flow defined by such an operator on $\widehat{K}$ preserves $\mathcal{L}^{fake}$, and therefore maps its tangent spaces to tangent spaces, and ruling spaces to ruling spaces, and moreover, the operators regular at $q=1$ preserve each ruling and tangent space. It follows that $(1-q)\mathbf{g}_{(1)}\in(1-q)T_{t'}\subset\mathcal{L}^{fake}$, where

$$T_{t'}:=e^{\sum_{k>0}\Psi^k(\epsilon D(Pq^{kQ\partial_Q},1))/k^2(1-q)}T_t.$$

(ii) We have

$$\Psi^m(T_{t'}) = e^{\sum_{k>0} \Psi^{mk}(\epsilon D(Pq^{kQ\partial_Q},1))/k^2(1-q^m)}\Psi^m(T_t).$$

On the other hand, for a primitive $m$th root of unity $\zeta$,

$$\begin{aligned}\mathbf{g}_{(\zeta)}(q^{1/m}/\zeta) &= e^{\sum_{k>0}\Psi^k(\epsilon D(P(q^{1/m}/\zeta)^{kQ\partial_Q},q^{1m}/\zeta))/k(1-q^{k/m}/\zeta^k)}\mathbf{f}_{(\zeta)}(q^{\mathbf{1}/m}/\zeta)\\ &= e^A\, e^{\sum_{l>0}\Psi^{ml}(\epsilon D(Pq^{lQ\partial_Q},1))/ml(1-q^l)}\mathbf{f}_{(\zeta)}(q^{1/m}/\zeta),\end{aligned}$$

where $A$ is some operator regular at $q=1$. It comes out of refactoring $e^{A+B/(1-q)}$, where $A$ and $B$ are regular at $q=1$, as $e^A e^{B/(1-q)}$. We use here the fact that the operators $A$ and $B$ have constant coefficients, and hence commute.

Note that the exponents $\sum_{k>0}\Psi^{mk}(\epsilon D(Q,Pq^{kQ\partial_Q},1))/k^2(1-q^m)$ and $\sum_{l>0}\Psi^{ml}(\epsilon D(Q,Pq^{lQ\partial_Q},1))/ml(1-q^l)$ agree modulo terms regular at $q=1$ (which, again, commute with the singular terms). Since we are given that

$$\mathbf{f}_{(\zeta)}(q^{1/m}/\zeta) \in \nabla_\zeta\, \Psi^m(T_t)\otimes_{\Psi^m(\Lambda)}\Lambda,$$

and since $\nabla_\zeta$ commutes with $\mathcal{D}_q$, we conclude (using the refactoring again), that

$$\mathbf{g}_\zeta(q^{1/m}/\zeta) \in \nabla_\zeta\, \Psi^m(T_{t'})\otimes_{\Psi^m(\Lambda)}\Lambda.$$

Note that the exponent in $e^A$ involves translations $P_i q^{Q_i\partial_{Q_i}}$ as well as $\zeta^{-Q_i\partial_{Q_i}}$, and so it is important, that (as we've checked in the proof of above Proposition), such operators preserve the space $\Psi^m(T_{t'})\otimes_{\Psi^m(\Lambda)}\Lambda$.

(iii) If $\mathbf{f}$ is regular at $q=1/\zeta$, where $\zeta\neq 0,\infty$ is not a root of unity, $\mathbf{g}$ is obviously regular there too. $\square$

`Corollary` (the $q$-string equation). *The range $\mathcal{L}\subset\mathcal{K}$ of permutation -equivariant J-function is invariant under the multiplication operators:*

$$\mathbf{f}\mapsto e^{\sum_{k>0}\Psi^k(\epsilon)/k(1-q^k)}\,\mathbf{f},\quad \epsilon\in\Lambda_+.$$

`Proof:` Use Theorem 1 with $D=1$.

## Examples

`Example 1:` $d=0$. In degree 0, i.e. modulo Novikov's variables, the cone $\mathcal{L}\subset\mathcal{K}$ coincides with the cone $\mathcal{L}_{pt}$ over the $\lambda$-algebra $K^0(X)\otimes\Lambda$. Theorem 1 and Proposition allow one to recover the part of $\mathcal{L}_{pt}$ over the $\lambda$-algebra $\Lambda' = K^0(X)_{pr}\otimes\Lambda$, where by $K^0(X)_{pr}$ (the *primitive* part) we denote the part of the ring $K^0(X)$ generated by line bundles.

Let monomials $P^a := P_1^{\alpha_1} \cdots P_K^{a_K}$ run a basis of $K^0(X)_{pr}$. Applying the above theorem to the finite difference operator

$$D = \sum_a \epsilon_a P^a q^{aQ\partial_Q} := \sum_a \epsilon_a \prod_{i=1}^K P_i^{a_i} q^{a_i Q_i \partial_{Q_i}}, \quad \epsilon_a \in \Lambda_+,$$

and acting on the point $\mathcal{J} \equiv 1-q$ *modulo Novikov's variables*, we recover over $\Lambda'$ the small J-function of the point:

$$(1-q)e^{\sum_a \sum_{k>0} \Psi^k(\epsilon_a)P^{ka}/k(1-q^k)} \equiv 1-q+\sum_a \epsilon_a P^a \mod \mathcal{K}_-.$$

Furthermore, applying linear combinations

$$\sum_a c_a(q) P^a q^{aQ\partial_Q}$$

with coefficients $c_a \in \Lambda[q, q^{-1}]$ which are arbitrary Laurent polynomials in $q$, we get, according to Proposition, points in the same ruling space of the cone $\mathcal{L}$. Modulo Novikov's variables this effectively results in multiplying by arbitrary elements $\sum_a c_a(q)P^a$ from $\Lambda'[q, q^{-1}]$, and therefore yields the entire cone $\mathcal{L}_{pt}$ over $\Lambda'$.

`Example 2:` $X = \mathbb{C}P^1$. We know[1] one point on $\mathcal{L} = \mathcal{L}_{\mathbb{C}P^1}$, the small J-function:

$$\mathcal{J}(0) = (1-q)\sum_{d\geq 0} \frac{Q^d}{(1-Pq)^2(1-Pq^2)^2\cdots(1-Pq^d)^2}.$$

Here $P = \mathcal{O}(-1)$ is the generator of $K^0(\mathbb{C}P^1)$. It satisfies the relation $(1-P)^2 = 0$. The K-theoretic Poincaré pairing is determined by

$$\chi(\mathbb{C}P^1; \phi(P)) = \mathrm{Res}_{P=1} \frac{\phi(P)}{(1-P)^2}\frac{dP}{P}.$$

We use Theorem 1 with the operator $D = \lambda + \epsilon P q^{Q\partial_Q}$, $\lambda, \epsilon \in \Lambda_+$, and obtain a 2-parametric family of points on $\mathcal{L}_{\mathbb{C}P^1}$:

$$e^{\sum_{k>0}(\Psi^k(\lambda)+\Psi^k(\epsilon)P^k q^{kQ\partial_Q})/k(1-q^k)}\mathcal{J}(0) =$$

$$(1-q)e^{\sum_{k>0}\Psi^k(\lambda)/k(1-q^k)} \sum_{d\geq 0} \frac{Q^d\, e^{\sum_{k>0}\Psi^k(\epsilon)P^k q^{kd}/k(1-q^k)}}{(1-Pq)^2(1-Pq^2)^2\cdots(1-Pq^d)^2}.$$

Examine now two specializations.

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[1]From various sources: Part IV (by localization), or [6] (by adelic characterization), or [5] (by toric compactifications).

Firstly, as a consistency check, let us extract from this the *degree*-1 *part of* $\mathcal{F}_0$. Modulo $Q^2$, we are left with

$$\mathcal{J} \equiv e^{\sum_{k>0}\Psi^k(\lambda)/k(1-q^k)}\left(1-q+\frac{(1-q)Q}{(1-Pq)^2}\, e^{\sum_{k>0}\Psi^k(\epsilon)P^kq^k/k(1-q^k)}\right).$$

Modulo $\mathcal{K}_-$ (and $Q^2$), we have: $[\mathcal{J}]_+ \equiv 1-q+\lambda+\epsilon P$. According to Part VII, Corollary 3,

$$\mathcal{F}_0(\mathbf{t}) = -\frac{1}{2}\Omega([\mathcal{J}(\mathbf{t})]_+, \mathcal{J}(\mathbf{t})) - \frac{1}{2}(\Psi^2(\mathbf{t}(1)),1).$$

For degree $d=1$ part $\mathcal{J}_1$ of $\mathcal{J}$, we have

$$-\Omega([\mathcal{J}]_+, \mathcal{J}_1) = \mathrm{Res}_{q=0,\infty}\left(1-\frac{1}{q}+\lambda+\epsilon P, \frac{(1-q)}{(1-Pq)^2}\, e^{A(q)}\right)\frac{dq}{q},$$

where $A(q) = \sum_{k>0}(\Psi^k(\lambda)+\Psi^k(\epsilon)P^kq^k)/k(1-q^k)$. The 1-form has no pole at $q=\infty$. Since $((1-q)/(1-Pq)^2)'_{q=0} = 2P-1$, and $A'(0) = \lambda+\epsilon P$, the residue at $q=0$ is calculated as

$$\begin{aligned}\left(1+\lambda+\epsilon P, e^{A(0)}\right) - \left(1, (2P-1)e^{A(0)} + (\lambda+\epsilon P)e^{A(0)}\right) = \\ \mathrm{Res}_{P=1}\frac{2(1-P)e^{A(0)}}{(1-P)^2}\frac{dP}{P} = 2e^{A(0)} = 2e^{\sum_{k>0}\Psi^k(\lambda)/k}.\end{aligned}$$

Let us check this rather trivial result "by hands". The degree $d=1$ part of $\mathcal{F}_0(t)$ at $t=\lambda+\epsilon P$ is defined as $\sum_{n\geq 0}\langle \lambda+\epsilon P, \dots, \lambda+\epsilon P\rangle_{0,n,1}^{S_n}$. Since there is only one rational curve of degree 1 in $\mathbb{C}P^1$, the moduli space $X_{0,n,1} = \overline{\mathcal{M}}_{0,n}(\mathbb{C}P^1, 1)$ is obtained from $(\mathbb{C}P^1)^n$ by some blow-ups along the diagonals. The evaluation maps $\mathrm{ev}_i : X_{0,n,1} \to \mathbb{C}P^1$ factor through $(\mathbb{C}P^1)^n$ as the projections $(\mathbb{C}P^1)^n \to \mathbb{C}P^1$. Therefore the correlator sum can be evaluated as

$$\sum_{n\geq 0}\left(H^*\left(\mathbb{C}P^1; \lambda+\epsilon P\right)^{\otimes n}\right)^{S_n} = \sum_{n\geq 0}(\lambda^{\otimes n})^{S_n}$$

because for $P=\mathcal{O}(-1)$ we have $H^*(\mathbb{C}P^1; P) = 0$. Let us remind from Part I that for elements of a $\lambda$-algebra,

$$(\lambda^{\otimes n})^{S_n} := \frac{1}{n!}\sum_{h\in S_n}\prod_{k>0}\Psi^{l_k(h)}(\lambda),$$

where $l_k(h)$ is the number of cycles of length $k$ in the permutation $h$. Thus, the correlator sum indeed coincides with $e^{\sum_{k>0}\Psi^k(\lambda)/k}$.

Secondly, let us return to our 2-parametric family of points on $\mathcal{L}_{\mathbb{C}P^1}$, and specialize it to the *symmetrized* theory, where only the $S_n$-invariant part of sheaf cohomology is taken into account. For this, we specialize the $\lambda$-algebra to $\Lambda = \mathbb{Q}[[\lambda, \epsilon, Q]]$ with $\Psi^k(\lambda) = \lambda^k, \Psi^k(\epsilon) = \epsilon^k$ (and $\Psi^k(Q) = Q^k$ as before). Some simplifications ensue. Since

$$q^{kd} = 1 - (1-q^k)(1+q^k+\cdots+q^{k(d-1)}),$$

we have

$$e^{\sum_{k>0}\epsilon^k P^k q^{kd}/k(1-q^k)} = e^{\sum_{k>0}\epsilon^k P^k/k(1-q^k)}\prod_{r=0}^{d-1} e^{-\sum_{k>0}\epsilon^k P^k q^{rk}/k}$$
$$= e^{\sum_{k>0}\epsilon^k P^k/k(1-q^k)}\prod_{r=0}^{d-1}(1-\epsilon P q^r).$$

Thus, we obtain the following 2-parametric family of points on $\mathcal{L}^{sym}_{\mathbb{C}P^1}$:

$$\mathcal{J}^{sym}_{\mathbb{C}P^1} = (1-q)e^{\sum_{k>0}(\lambda^k+\epsilon^k P^k)/k(1-q^k)}\sum_{d\geq 0} Q^d \frac{\prod_{r=0}^{d-1}(1-\epsilon P q^r)}{\prod_{r=1}^{d}(1-Pq^r)^2}.$$

Note that the projection of this series to $\mathcal{K}_+$ along $\mathcal{K}_-$ picks contributions only from the terms with $d=0$ and $k=1$:

$$[\mathcal{J}^{sym}_{\mathbb{C}P^1}]_+ = 1-q+\lambda+\epsilon P.$$

Therefore the series represents *the small J-function of the symmetrized quantum K-theory of* $\mathbb{C}P^1$. The exponential factor is actually equal to $\exp_q(\lambda/(1-q))\exp_q(\epsilon P/(1-q))$. Thus, we obtain:

$$\mathcal{J}^{sym}_{\mathbb{C}P^1}(\lambda+\epsilon P) =_{\mod (1-P)^2}$$
$$(1-q)\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{d=0}^{\infty}\frac{\lambda^m\,\epsilon^l\,P^l\,Q^d\,\prod_{r=0}^{d-1}(1-\epsilon P q^r)}{\prod_{t=1}^{m}(1-q^t)\prod_{s=1}^{l}(1-q^s)\prod_{r=1}^{d}(1-Pq^r)}.$$

## Reconstruction theorems

As in Example 1, assume that $p_1,\dots,p_K$ is a numerically effective integer basis in $H^2(X,\mathbb{Q})$, that Novikov's monomials $Q^d = Q_1^{d_1}\dots Q_K^{d_K}$ represent degree $d$ holomorphic curves in $X$ in coordinates $d_i = p_i(d)$ on $H_2(X)$, that $P_i$ are line bundles with $\operatorname{ch} P_i = e^{-p_i}$, and that monomials $P^a = P_1^{a_1}\cdots P_K^{a_k}$ run a basis in $K^0(X)_{pr}$, the primitive part of the K-ring. We also write $a.d$ for the value $\sum a_i d_i$ of $-c_1(P^a)$ on $d$.

`Theorem 2` (explicit reconstruction). *Let $I=\sum_d I_d Q^d$ be a point in the range $\mathcal{L}\subset\mathcal{K}$ of the J-function of permutation-equivariant quantum K-theory on $X$, written as a vector-valued series in Novikov's variables. Then the following family also lies in $\mathcal{L}$:*

$$\sum_d I_d Q^d e^{\sum_{k>0}\sum_a \Psi^k(\epsilon_a)P^{ka}q^{k(a.d)}/k(1-q^k)},\quad \epsilon_a\in\Lambda_+.$$

*Moreover, for arbitrary Laurent polynomials $c_a\in\Lambda[q,q^{-1}]$, the following series also lies in $\mathcal{L}$:*

$$\sum_d I_d Q^d e^{\sum_{k>0}\sum_a \Psi^k(\epsilon_a)P^{ka}q^{k(a.d)}/k(1-q^k)}\sum_a c_a(q)P^a q^{a.d}.$$

*Furthermore, when $K^0(X)=K^0(X)_{pr}$, the whole cone $\mathcal{L}\subset\mathcal{K}$ is parameterized this way.*

`Proof.` We first work over $\Lambda'$ freely generated as $\lambda$-algebra by the "time" variables $\epsilon_a$, and use Theorem 1 with the $Q$-independent finite difference operator $D=\sum_a \epsilon_a P^a q^{aQ\partial_Q}$. We conclude that the family

$$e^{\sum_{k>0}\sum_a \Psi^k(\epsilon_a)P^{ka}q^{kaQ\partial_Q}/k(1-q^k)}I=\sum_d I_d Q^d e^{\sum_{k>0}\sum_a \Psi^k(\epsilon_a)P^a q^{k(a.d)}/k(1-q^k)}$$

lies in the cone $\mathcal{L}$, defined over $\Lambda'$. To obtain the second statement, we apply Proposition, using finite difference operators $\sum_a c_a(q)P_a q^{aQ\partial_Q}$. Afterwards we specialize the "times" $\epsilon_a$ to any values $\epsilon_a\in\Lambda$ (which at this point may become dependent on $Q$). Finally, when $K^0(X)=K^0(X)_{pr}$, we use the formal Implicit Function Theorem to conclude that the whole cone $\mathcal{L}$ is parameterized, because this is true modulo Novikov's variables, as Example 1 shows. □

`Example:` $X=\mathbb{C}P^N$. According to Theorem 2, the entire cone $\mathcal{L}$ is parameterized as follows:

$$\mathcal{J}=(1-q)\sum_{d\geq 0}Q^d\,\frac{e^{\sum_{k>0}\sum_{a=0}^N \Psi^k(\epsilon_a)P^{ka}q^{kad}/k(1-q^k)}\sum_{a=0}^N c_a(q)P^a q^{ad}}{(1-Pq)^{N+1}(1-Pq^2)^{N+1}\cdots(1-Pq^d)^{N+1}}.$$

Of course, this is obtained by applying Theorem 2 to the small J-function $\mathcal{J}(0)$ from [5] (also [6], or Parts II–IV in the non-equivariant limit). Here $\epsilon_a\in\Lambda$, $c_a(q)$ are arbitrary Laurent polynomials in $q$ with coefficients in $\Lambda$, and $P^a$, $a=0,\dots,N$, $P=\mathcal{O}(-1)$, are used for a basis in $K^0(X)$. Perhaps, the basis $(1-P)^a, a=0,\dots,N$, is more useful (cf. [4]), and we get yet another parameterization of $\mathcal{L}$:

$$(1-q)\sum_{d\geq 0}Q^d\,\frac{e^{\sum_{k>0}\sum_{a=0}^N \Psi^k(\epsilon_a)(1-P^kq^{kd})^a/k(1-q^k)}\sum_{a=0}^N c_a(q)(1-Pq^d)^a}{(1-Pq)^{N+1}(1-Pq^2)^{N+1}\cdots(1-Pq^d)^{N+1}}.$$

We return now to the context of Part VII, where we studied the mixed J-function $\mathcal{J}(\mathbf{x},\mathbf{t})$, involving two types of inputs: permutable $\mathbf{t}$ and non-permutable $\mathbf{x}$, both taken from $\mathcal{K}_+$. The cone $\mathcal{L}\subset\mathcal{K}$ represents the range of $\mathbf{t}\mapsto\mathcal{J}(0,\mathbf{t})$. Recall that according to the general theory, it is the union of *ruling spaces* $(1-q)T_t$, where $t=\mathcal{T}(\mathbf{t})$ is given by a certain non-linear map

$$\mathcal{T}: K^0(X)\otimes\Lambda_+\oplus(1-q)\mathcal{K}_+\to K^0(X)\otimes\Lambda_+.$$

At the same time, for a fixed value of $\mathbf{t}$, the range of the *ordinary* J-function $\mathbf{x}\mapsto\mathcal{J}(\mathbf{x},\mathbf{t})$ is an overruled Lagrangian cone $\mathcal{L}_t\subset\mathcal{K}$, which shares with $\mathcal{L}$ one ruling space, $T_t$, corresponding to $t=\mathcal{T}(\mathbf{t})$. Each tangent space of each cone $\mathcal{L}_t$ is tangent to $\mathcal{L}_t$ along one of the ruling spaces (e.g. $T_t$ is tangent along $(1-q)T_t$), and is related with this ruling space by the multiplication by $1-q$. As a consequence, not only each ruling (and tangent) space of each $\mathcal{L}_t$ is a $D_q$-module (which is proved on the basis of adelic characterization as in Proposition above), but also each cone $\mathcal{L}_t$ is invariant under the flow

$$\mathbf{f}\mapsto e^{\epsilon D(Q,Pq^{Q\partial_Q},q)/(1-q)}\mathbf{f},$$

where $D\in\mathcal{D}_q$. We use this to reconstruct the family $\mathcal{L}_t$.

`Theorem 3.` *Let $I=\sum I_dQ^d$ (as in Theorem 2). Then*

$$I(\epsilon)=\sum_d I_d(\epsilon)Q^d:=\sum_d I_dQ^de^{\sum_{k>0}\sum_a\Psi^k(\epsilon_a)P^{ka}q^{k(a.d)}/k(1-q^k)},\quad \epsilon_a\in\Lambda_+$$

*represent a family of points on the cones $\mathcal{L}_{t(\epsilon)}$ (one point on each cone), and the following family of points, parameterized by $\tau_a\in\Lambda$ and by $c_a\in\Lambda[q,q^{-1}]$, lies on $\mathcal{L}_{t(\epsilon)}$:*

$$\sum_d I_d(\epsilon)\,Q^d\,e^{\sum_a\tau_aP^aq^{a.d}}/(1-q)\sum_a c_a(q)P^aq^{a.d}.$$

*Moreover, if $K^0(X)=K^0(X)_{pr}$, for each $t\in K^0(X)\otimes\Lambda_+$ the whole cone $\mathcal{L}_t$ is thus parameterized.*

`Proof.` It is clear from computation modulo Novikov's variables that the family $I(\epsilon)$ has no tangency with the ruling spaces, hence represents at most one point from each $\mathcal{L}_t$ (and does represent one, when $K^0(X)=K^0(X)_{pr}$). Given one point, $I(\epsilon)$, on $\mathcal{L}_{t(\epsilon)}$, we generate more points by machinery discussed above: applying the commuting flows

$$e^{\sum_a\tau_aP^aq^{O\partial_Q}/(1-q)}I(\epsilon)=\sum_d Q^dI_d(\epsilon)\,e^{\sum_a\tau_aP^aq^{a.d}/(1-q)},$$

followed by the application of the operators $\sum_a c_a(q) P^q q^{aQ\partial_Q}$, where $\tau_a$ and the coefficients of $c_a$ are independent variables. Afterwards they can be specialized to some values in $\Lambda$ (in particular, depending on $Q$). In the case when $K^0(X) = K^0(X)_{pr}$, it follows from the Implicit Function Theorem and Example 1 about the limit to $d = 0$, that the entire cone $\mathcal{L}_t$ for each $t$ is thus obtained. $\square$

`Example:` $X = \mathbb{C}P^N$. It follows that for fixed values of $\epsilon_a$, the corresponding cone $\mathcal{L}_{t(\epsilon)}$ is parameterized as

$$(1-q)\sum_{d\geq 0} Q^d \, \frac{e^{\sum_{a=0}^N \left(\tau_a \frac{P^a q^{ad}}{1-q} + \sum_{k>0} \Psi^k(\epsilon_a) \frac{P^{ka} q^{kad}}{k(1-q^k)}\right)} \sum_{a=0}^N c_a(q) P^a q^{ad}}{(1-Pq)^{N+1}(1-Pq^2)^{N+1}\cdots(1-Pq^d)^{N+1}},$$

and all $\mathcal{L}_t$ are so obtained.

`Remarks.` Reconstruction theorems in quantum cohomology and K-theory go back to Kontsevich–Manin [9] and Lee–Pandharipande [10] respectively. Theorem 3 is a slight generalization (from the case $t = 0$) of the "explicit reconstruction" result [4] in the ordinary (non-permutation-equivariant) quantum K-theory, which in its turn mimics the results of quantum cohomology theory already found in [1, 7], and shares the methods based on finite difference operators with the K-theoretic results of [8].

Theorems of this section show that when $K^0(X)$ is generated by line bundles, the entire range $\mathcal{L}$ of the J-function in the permutation-equivariant genus-0 quantum K-theory of $X$, as well as the entire family $\mathcal{L}_t$ of the overruled Lagrangian cones representing the "ordinary" J-functions, depending on the permutable parameter, $t$, can be explicitly represented in a parametric form, *given one point on any of these cones.* In essence, all genus-0 K-theoretic GW-invariants of $X$, permutation-equivariant, ordinary, or mixed, are thereby reconstructed from any one point: a $K^0(X)$-valued series $\sum_d I_d Q^d$ in Novikov's variables.

In the case of a toric $X$, the results of Part V exhibit such a point in the form of the *q-hypergeometric series* mirror-symmetric to $X$. Needless to say, the same applies to toric bundles spaces, or super-bundles (a.k.a. toric complete intersections), as well as to the torus-equivariant versions of K-theoretic GW-invariants. Thus "all" (torus-equivariant or not; permutation-equivariant, ordinary, or mixed) K-theoretic genus-0 GW-invariants of toric manifolds, toric bundles, or toric complete intersections are computed in a geometrically explicit form, illustrated by the above example.

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