PERMUTATION-EQUIVARIANT
QUANTUM K-THEORY VIII.
EXPLICIT RECONSTRUCTION

ALEXANDER GIVENTAL

ABSTRACT. In Part VII, we proved that the range $\mathcal{L}_X$ of the big J-function in permutation-equivariant genus-0 quantum K-theory is an overruled cone, and gave its adelic characterization. Here we show that the ruling spaces are $D_q$-modules in Novikov’s variables, and moreover, that the whole cone $\mathcal{L}_X$ is invariant under a large group of symmetries of $\mathcal{L}_X$ defined in terms of $q$-difference operators. We employ this for the explicit reconstruction of $\mathcal{L}_X$ from one point on it, and apply the result to toric $X$, when such a point is given by the $q$-hypergeometric function.

ADELIC CHARACTERIZATION

We begin where we left in Part VII: at a description of the range $\mathcal{L} \subset \mathcal{K}$ in the space $\mathcal{K}$ of $K^0(X) \otimes \Lambda$-value rational functions of $q$ of the J-function of permutation-equivariant quantum K-theory of a given Kähler target space $X$:

$$J := 1 - q + t(q) + \sum_{\alpha} \phi_\alpha \sum_{n,d} Q^d (\frac{\phi^\alpha}{1 - qL}; t(L), \ldots, t(L)) S_n_{0,1+n,d}.$$ 

We proved that $\mathcal{L}$ is an overruled cone, i.e. it is swept by a family of certain $\Lambda[q, q^{-1}]$-modules, called ruling spaces:

$$\mathcal{L} = \bigcup_{t \in \Lambda_+} (1 - q) S(t)^{-1} \mathcal{K}_+,$$

where $S_t$ is a certain family of “matrix” functions rational in $q$, whose construction we are not going to remind here. Let us recall the adelic characterization of $\mathcal{L}$, which will be our main technical tool.

It is given in terms of the overruled cone $\mathcal{L}^{fake} \subset \hat{\mathcal{K}}$ in the space of vector-valued Laurent series in $q - 1$, describing the range of the

Date: August 8, 2015.

This material is based upon work supported by the National Science Foundation under Grant DMS-1007164, and by the IBS Center for Geometry and Physics, POSTECH, Korea.
J-function in *fake* quantum K-theory of $X$:

$$L^\text{fake} = \bigcup_{t \in \Lambda} (1 - q)T_t, \quad T_t = S^\text{fake}(q_t)^{-1}\hat{K}_+.$$  

Here $S^\text{fake}_t$ is some “matrix” Laurent series, $T_t$ is a tangent space to $L^\text{fake}$, containing $(1 - q)T_t$, and tangent to $L^\text{fake}$ at all points of the ruling space $(1 - q)T_t$.

According to the last section of Part VII, a rational function $f \in \mathcal{K}$ lies in $L$ if and only if its Laurent series expansions $f_\zeta$ near $q = 1/\zeta$ satisfy the following three conditions:

(i) $f_1 \in L^\text{fake}$;

(ii) when $\zeta \neq 0, 1, \infty$ is a primitive $m$th root of unity,

$$f_\zeta(q^{1/m}/\zeta) \in L^\zeta,$$

a certain subspace in $\hat{K}$, determined by the tangent space $T_t$ to $L^\text{fake}$ at the point $f_1$;

(iii) when $\zeta \neq 0, \infty$ is not a root of unity, $f_\zeta$ is a power series in $q - 1/\zeta$, i.e. $f$ has no pole at $q = 1/\zeta$.

The subspace $L^\zeta$ is described as $\nabla_\zeta\Psi^m(T_t) \otimes \Psi^m(\Lambda)$, where the Adams operation $\Psi^m$ acts by $\Psi^m(q) = q^m$ and naturally on the $\lambda$-algebra $K^0(X) \otimes \Lambda$, and $\nabla_\zeta$ is the operator of multiplication by

$$e^{\sum_{k > 0} \left( \frac{\Psi^k(T_X)}{k(1 - \zeta^{1/q^k})} - \frac{\Psi^k(T_X)}{k(1 - q^k)} \right)}.$$

In its turn, the cone $L^\text{fake} \subset \hat{K}$ (and hence its tangent spaces $T_t$) can be expressed in terms of the cone $L^H \subset H$, describing the range of cohomological J-function in the space $H$ of Laurent series in one indeterminate $z$ with coefficients in $H_{\text{even}}(X) \otimes \Lambda$. Namely, according to the Hirzebruch–Riemann–Roch theorem [2] in fake quantum K-theory,

$$\text{qch}(L^\text{fake}) = \Delta L^H,$$

where the quantum Chern character $\text{qch} : \hat{K} \to H$ acts by $\text{qch} q = e^z$, and by the natural Chern character $\text{ch} : K^0(X) \otimes \Lambda \to H_{\text{even}}(X) \otimes \Lambda$ on the vector coefficients, while $\Delta$ acts as the multiplication in the classical cohomology of $X$ by the Euler–Maclaurin asymptotics (see [3, 2, 6]) of the infinite product:

$$\Delta \sim \prod_{r=1}^{\infty} \text{td}(T_X \otimes q^{-r}).$$

Using all these descriptions, we are going on explore how the *string and divisor equations* of quantum cohomology theory manifest in the genus-0 permutation-equivariant quantum K-theory.
DIVISOR EQUATIONS AND $\mathcal{D}_q$-MODULES

Let $p_1, \ldots, p_K$ be a basis in $H^2(X, \mathbb{R})$ consisting of integer numerically effective classes, and let $Q^d = Q_1^{d_1} \cdots Q_K^{d_K}$, where $d_i = p_i(d)$, represent degree-$d$ holomorphic curves in the Novikov ring. We remind that the Novikov variables are included into the ground algebra $\Lambda$.

The loop space $\mathcal{H}$ of Laurent series in $z$ with vector coefficients in $H^{even}(X) \otimes \Lambda$ is equipped with the structure of a module over the algebra $\mathcal{D}$ of differential operators in the Novikov variables, so that $Q_i$ acts as multiplication by $Q_i$, and $Q_i\partial Q_i$ acts as $zQ_i\partial Q_i - p_i$. The divisor equations in quantum cohomology theory imply (see e.g. [4]) that

linear vector fields $f \mapsto (Q_i \partial Q_i - p_i/z)f$ in $\mathcal{H}$ are tangent to $\mathcal{L}^H \subset \mathcal{H}$.

In follows that the ruling spaces (as well as tangent spaces) of $\mathcal{L}^H$ are $\mathcal{D}$-modules, i.e. are invariant with respect to each differential operator $D(Q, zQ\partial Q - p, z)$, and moreover the flow $\epsilon \mapsto e^{\epsilon D/\epsilon}$ of the vector field $f \mapsto DF/z$ preserves $\mathcal{L}^H$.

Indeed, for $f \in \mathcal{L}^H$, the vector $(Q_i \partial Q_i - p_i)f$ lies in $T_f \mathcal{L}^H$, and hence $(zO_i \partial Q_i - p_i)f$ lies in the same ruling space $zT_f \mathcal{L}^H$ as $f$ does. Therefore so does $DF$, and hence $DF/z \in T_f \mathcal{L}^H$, i.e. the vector field $f \mapsto DF/z$ is tangent to $\mathcal{L}^H$.

Note that the operator $\triangle$ relating $\mathcal{L}^H$ and $\mathcal{L}^{fake}$ involves multiplication in the commutative classical cohomology algebra $H^{even}(X)$, but does not involve Novikov’s variables. Consequently, the tangent and ruling spaces of $\text{qch}(\mathcal{L}^{fake})$ are $\mathcal{D}$-modules too, and moreover, the flows $\epsilon \mapsto e^{\epsilon D/\epsilon}$ preserve $\text{qch}(\mathcal{L}^{fake})$.

We equip the space $\mathcal{K}$ of vector-valued rational functions of $q$ with the structure of a module over the algebra $\mathcal{D}_q$ of finite difference operators. It is generated (over the algebra of Laurent polynomials in $q$) by multiplication operators, acting as multiplications by $Q_i$, and translation operators, acting as $P_i q^{Q_i \partial Q_i}$, where $P_i$ is the multiplication in $K^0(X)$ by the line bundle with the Chern character $ch P_i = e^{-p_i}$.

**Proposition** (cf. [6, 4]). The ruling spaces of the overruled cone $\mathcal{L} \subset \mathcal{K}$ of permutation-equivariant quantum $K$-theory are $\mathcal{D}_q$-modules.

**Proof.** If $f \in \mathcal{L}$, it passes the tests (i),(ii),(iii) of adelic characterization. We need to show that $g := P_i q^{Q_i \partial Q_i} f$, which obviously lies in $\mathcal{K}$, also passes the tests (and with the same $t \in K^0(X) \otimes \Lambda_+$). This is obvious for test (iii), and is true about test (i) because of the above $\mathcal{D}$-module (and hence $\mathcal{D}_q$-module) property of the ruling spaces $(1 - q)T_t$ of $\mathcal{L}^{fake}$. To verify test (ii), we write:

$$g(q^{1/m}/\zeta) = P_i(q^{1/m})Q_i \partial Q_i \cdot q^{1/m}, f(q^{1/m}/\zeta).$$
First, note that the operator $\nabla_\zeta$ relating $L(\zeta)$ with $\Psi^m(T_t)$ does not involve Novikov’s variables and commutes with $D_q$. Next, let us elucidate the notation $\Psi^m(T_t) \otimes \Psi^m(\Lambda)$. In fact the space so indicated consists of linear combinations $\sum_a \lambda_a(Q, q)\Psi^m(f_a)$, where $f_a \in T_t$, and $\lambda_a \in \Lambda[[q - 1]]$. We have the following commutation relations:

$$P_i^m \Psi^m = \Psi^m P_i^m, \quad (q^{1/m})Q_i \partial Q_i \Psi^m = q^{Q_i^m \partial Q_i^m} \Psi^m = \Psi^m(q^{1/m})Q_i \partial Q_i,$$

$$\zeta Q_i \partial Q_i \Psi^m = \zeta Q_i^m \partial Q_i^m \Psi^m = \Psi^m.$$ 

Therefore

$$P_i(q^{1/m})Q_i \partial Q_i \Psi^m \left( \sum_a \lambda_a \Psi^m(f_a) \right) = \sum_a (q^{1/m}/Q_i) Q_i \partial Q_i \Psi^m \left( P_i^m(q^{1/m})Q_i \partial Q_i f_a \right),$$

which lies in $\Psi^m(T_t) \otimes \Psi^m(\Lambda)$ since $T_t$ is invariant under the operator $P_i^m(q^{1/m})Q_i \partial Q_i = e(zQ_i \partial Q_i - p_i)/m$. □

Let $D(P q^{Q_0}, q)$ be a constant coefficient finite difference operator, by which we mean a Laurent polynomial expression in translation operators $P_i^m Q_i \partial Q_i$, and maybe $q$, with coefficients from $\Lambda$ independent of $Q$. We assume below that $\epsilon \in \Lambda_+$ to assure $\epsilon$-adic convergence of infinite sums.

**Theorem 1.** The operator

$$e \sum_{k>0} \Psi^k(\epsilon D(P q^{Q_0}, q))/k(1 - q^k)$$

preserves $L \subset K$.

**Proof.** We show that if $(1 - q)f$ passes tests (i), (ii), (iii) of the adelic characterization of $L$, then $(1 - q)g$, where

$$g := e \sum_{k>0} \Psi^k(\epsilon D(P q^{Q_0}, q))/k(1 - q^k) f,$$

also does.

(i) Suppose $(1 - q)f(1)$ lies in the ruling space $(1 - q)T_t \subset L^{fake}$. Note that the exponent $\sum_{k>0} \Psi^k(\epsilon D(P q^{Q_0}, q))/k(1 - q^k)$ has first order pole at $q = 1$. According to the discussion above the flow defined by such an operator on $\hat{K}$ preserves $L^{fake}$, and therefore maps its tangent spaces to tangent spaces, and ruling spaces to ruling spaces, and moreover, the operators regular at $q = 1$ preserve each ruling and tangent space. It follows that $(1 - q)g(1) \in (1 - q)T_t' \subset L^{fake}$, where

$$T_t' := e \sum_{k>0} \Psi^k(\epsilon D(P q^{Q_0}, 1))/k^2(1 - q)T_t.$$
(ii) We have
\[ \Psi^m(T'_t) = e \sum_{k>0} \Psi^{mk}(\epsilon D(Pq^{kQ\partial_Q}, 1))/k^2(1 - q^m) \Psi^m(T_t). \]
On the other hand, for a primitive \( m \)th root of unity \( \zeta \),
\[ g(\zeta)(q^{1/m}/\zeta) = e \sum_{k>0} \Psi^{mk}(\epsilon D(Pq^{kQ\partial_Q}, 1))/k(1 - q^k/m) \Psi^m(T'_t) = e^A e \sum_{l>0} \Psi^{ml}(\epsilon D(Pq^{lQ\partial_Q}, 1))/ml(1 - q^l) \Psi^m(T'_t), \]
where \( A \) is some operator regular at \( q = 1 \). It comes out of refactoring
\( e^A + B/(1 - q) \), where \( A \) and \( B \) are regular at \( q = 1 \), as \( e^A e^B/(1 - q) \). We use here the fact that the operators \( A \) and \( B \) have constant coefficients, and hence commute.
Note that the exponents \( \sum_{k>0} \Psi^{mk}(\epsilon D(Q, Pq^{kQ\partial_Q}, 1))/k^2(1 - q^m) \) and \( \sum_{l>0} \Psi^{ml}(\epsilon D(Q, Pq^{lQ\partial_Q}, 1))/ml(1 - q^l) \) agree modulo terms regular at \( q = 1 \) (which, again, commute with the singular terms). Since we are given that
\[ f(\zeta)(q^{1/m}/\zeta) \in \nabla_\zeta \Psi^m(T_t) \otimes_{\Psi^m(\Lambda)} \Lambda, \]
and since \( \nabla_\zeta \) commutes with \( D_q \), we conclude (using the refactoring again), that
\[ g(\zeta)(q^{1/m}/\zeta) \in \nabla_\zeta \Psi^m(T'_t) \otimes_{\Psi^m(\Lambda)} \Lambda. \]
Note that the exponent in \( e^A \) involves translations \( P_iq^{iQ\partial_Q} \) as well as \( \zeta^{-Q_i\partial_Q} \), and so it is important, that (as we’ve checked in the proof of above Proposition), such operators preserve the space \( \Psi^m(T'_t) \otimes_{\Psi^m(\Lambda)} \Lambda. \)
(iii) If \( f \) is regular at \( q = 1/\zeta \), where \( \zeta \neq 0, \infty \) is not a root of unity, \( g \) is obviously regular there too. \( \square \)

**Corollary** (the \( q \)-string equation). *The range \( \mathcal{L} \subset \mathcal{K} \) of permutation-equivariant \( J \)-function is invariant under the multiplication operators:*
\[ f \mapsto e \sum_{k>0} \Psi^k(\epsilon)/(1 - q^k) f, \quad \epsilon \in \Lambda_. \]

**Proof:** Use Theorem 1 with \( D = 1 \).

**Examples**

**Example 1:** \( d = 0 \). In degree 0, i.e. modulo Novikov’s variables, the cone \( \mathcal{L} \subset \mathcal{K} \) coincides with the cone \( \mathcal{L}_{pr} \) over the \( \lambda \)-algebra \( K^0(X) \otimes \Lambda \). Theorem 1 and Proposition allow one to recover the part of \( \mathcal{L}_{pr} \) over the \( \lambda \)-algebra \( \Lambda' = K^0(X)_{pr} \otimes \Lambda \), where by \( K^0(X)_{pr} \) (the *primitive part*) we denote the part of the ring \( K^0(X) \) generated by line bundles.
Let monomials $P^a := P_1^{a_1} \cdots P_K^{a_K}$ run a basis of $K^0(X)_{pr}$. Applying the above theorem to the finite difference operator

$$D = \sum_a \epsilon_a P^a q^{aQ\partial_Q} := \sum_a \epsilon_a \prod_{i=1}^K P_i^{a_i} q^{a_i Q_i \partial_{Q_i}}, \quad \epsilon_a \in \Lambda_+,$$

and acting on the point $J \equiv 1 - q$ modulo Novikov's variables, we recover over $\Lambda'$ the small $J$-function of the point:

$$(1 - q) e^{\sum_a \sum_{k>0} \Psi^k(\epsilon_a) P^a k / k (1 - q^k)} \equiv 1 - q + \sum_a \epsilon_a P^a \mod K_-.$$

Furthermore, applying linear combinations \[\sum_a c_a(q) P^a q^{aQ\partial_Q}\] with coefficients $c_a \in \Lambda[q, q^{-1}]$ which are arbitrary Laurent polynomials in $q$, we get, according to Proposition, points in the same ruling space of the cone $L$. Modulo Novikov's variables this effectively results in multiplying by arbitrary elements $\sum_a c_a(q) P^a$ from $\Lambda'[q, q^{-1}]$, and therefore yields the entire cone $L_{pt}$ over $\Lambda'$.

**Example 2:** $X = \mathbb{C}P^1$. We know\(^1\) one point on $L = L_{\mathbb{C}P^1}$, the small $J$-function:

$$J(0) = (1 - q) \sum_{d \geq 0} \frac{Q^d}{(1 - P q)^2 (1 - P q^2)^2 \cdots (1 - P q^d)^2}.$$ 

Here $P = \mathcal{O}(-1)$ is the generator of $K^0(\mathbb{C}P^1)$. It satisfies the relation $(1 - P)^2 = 0$. The K-theoretic Poincaré pairing is determined by

$$\chi(\mathbb{C}P^1, \phi(P)) = \text{Res}_{P=1} \phi(P) \frac{dP}{(1 - P)^2 P}.$$ 

We use Theorem 1 with the operator $D = \lambda + \epsilon P q^{Q\partial_Q}$, $\lambda, \epsilon \in \Lambda_+$, and obtain a 2-parametric family of points on $L_{\mathbb{C}P^1}$:

$$(1 - q) e^{\sum_{k>0} \Psi^k(\lambda) + \Psi^k(\epsilon) P^k q^{kQ\partial_Q} / k (1 - q^k)} J(0) = (1 - q) e^{\sum_{k>0} \Psi^k(\lambda) / k (1 - q^k)} \sum_{d \geq 0} \frac{Q^d}{(1 - P q)^2 (1 - P q^2)^2 \cdots (1 - P q^d)^2}$$

Examine now two specializations.

---

\(^1\)From various sources: Part IV (by localization), or [6] (by adelic characterization), or [5] (by toric compactifications).
For degree 1 part of $F_0$. Modulo $Q^2$, we are left with

$$J \equiv e^{\sum_{k>0} \Psi^k(\lambda)/k(1-q^k)} \left(1 - q + \frac{(1-q)Q}{(1-Pq)^2} e^{\sum_{k>0} \Psi^k(e)P^kq^k/1-q^k} \right).$$

Modulo $K_0$ (and $Q^2$), we have: $[J]_+ \equiv 1 - q + \lambda + \epsilon P$. According to Part VII, Corollary 3,

$$F_0(t) = -\frac{1}{2} \Omega([J]_+,J(t)) - \frac{1}{2} (\psi^2(t(1)),1).$$

For degree $d=1$ part $J_1$ of $J$, we have

$$-\Omega([J]_+,J_1) = \text{Res}_{q=0,\infty} \left(1 - \frac{1}{q} + \lambda + \epsilon P, \frac{(1-q)Q}{(1-Pq)^2} e^{A(q)} \right) dq,$$

where $A(q) = \sum_{k>0} (\Psi^k(\lambda) + \Psi^k(e)P^kq^k)/k(1-q^k)$. The 1-form has no pole at $q = \infty$. Since $((1-q)/(1-Pq)^2)'_{q=0} = 2P - 1$, and $A'(0) = \lambda + \epsilon P$, the residue at $q = 0$ is calculated as

$$\text{Res}_{P=1} \frac{2(1-P)e^{A(0)} dP}{(1-P)^2} = 2e^{A(0)} = 2e^{\sum_{k>0} \Psi^k(\lambda)/k}.$$

Let us check this rather trivial result “by hands”. The degree $d=1$ part of $F_0(t)$ at $t = \lambda + \epsilon P$ is defined as $\sum_{n\geq0} (\lambda + \epsilon P, \ldots, \lambda + \epsilon P) S_n$. Since there is only one rational curve of degree 1 in $\mathbb{C}P^1$, the moduli space $X_{0,n,1} = \mathcal{M}_{0,n}(\mathbb{C}P^1,1)$ is obtained from $(\mathbb{C}P^1)^n$ by some blow-ups along the diagonals. The evaluation maps $ev_i : X_{0,n,1} \to \mathbb{C}P^1$ factor through $(\mathbb{C}P^1)^n$ as the projections $(\mathbb{C}P^1)^n \to \mathbb{C}P^1$. Therefore the correlator sum can be evaluated as

$$\sum_{n\geq0} \left( H^* (\mathbb{C}P^1; \lambda + \epsilon P) \otimes_n S_n \right) = \sum_{n\geq0} (\lambda^{\otimes n}) S_n$$

because for $P = O(-1)$ we have $H^* (\mathbb{C}P^1; P) = 0$. Let us remind from Part I that for elements of a $\lambda$-algebra,

$$(\lambda^{\otimes n}) S_n := \frac{1}{n!} \sum_{h\in S_n} \prod_{k>0} \Psi^k(\lambda),$$

where $l_k(h)$ is the number of cycles of length $k$ in the permutation $h$. Thus, the correlator sum indeed coincides with $e^{\sum_{k>0} \Psi^k(\lambda)/k}$. 

Firstly, as a consistency check, let us extract from this the degree-1 part of $F_0$. Modulo $Q^2$, we are left with

$$J \equiv e^{\sum_{k>0} \Psi^k(\lambda)/k(1-q^k)} \left(1 - q + \frac{(1-q)Q}{(1-Pq)^2} e^{\sum_{k>0} \Psi^k(e)P^kq^k/1-q^k} \right).$$

Modulo $K_0$ (and $Q^2$), we have: $[J]_+ \equiv 1 - q + \lambda + \epsilon P$. According to Part VII, Corollary 3,
Secondly, let us return to our 2-parametric family of points on \( \mathcal{L}_{\mathbb{C}P^1} \), and specialize it to the *symmetrized* theory, where only the \( S_n \)-invariant part of sheaf cohomology is taken into account. For this, we specialize the \( \lambda \)-algebra to \( \Lambda = \mathbb{Q}[\lambda, \epsilon, Q] \) with \( \Psi^k(\lambda) = \lambda^k \), \( \Psi^k(\epsilon) = \epsilon^k \) (and \( \Psi^k(Q) = Q^k \) as before). Some simplifications ensue. Since 

\[
q^{kd} = 1 - (1 - q^k)(1 + q^k + \cdots + q^{(d-1)}k),
\]

we have 

\[
e\sum_{k>0} \epsilon^k P^k q^{kd}/k(1 - q^k) = e\sum_{k>0} \epsilon^k P^k/k(1 - q^k) \prod_{r=0}^{d-1} e^{-\sum_{k>0} \epsilon^k P^k q^{rk}/k}
\]

\[
= e\sum_{k>0} \epsilon^k P^k/k(1 - q^k) \prod_{r=0}^{d-1} (1 - \epsilon P q^r).
\]

Thus, we obtain the following 2-parametric family of points on \( \mathcal{L}_{\mathbb{C}P^1}^{\text{sym}} \):

\[
\mathcal{J}_{\mathbb{C}P^1}^{\text{sym}} = (1 - q)e\sum_{k>0} (\lambda^k + \epsilon^k P^k)/k(1 - q^k) \sum_{d \geq 0} Q^d \prod_{r=0}^{d-1} (1 - \epsilon P q^r) \prod_{r=1}^{d} (1 - q^r)^2.
\]

Note that the projection of this series to \( \mathcal{K}_+ \) along \( \mathcal{K}_- \) picks contributions only from the terms with \( d = 0 \) and \( k = 1 \):

\[
[\mathcal{J}_{\mathbb{C}P^1}^{\text{sym}}]_+ = 1 - q + \lambda + \epsilon P.
\]

Therefore the series represents the *small J-function of the symmetrized quantum K-theory of \( \mathbb{C}P^1 \).* The exponential factor is actually equal to \( \exp_q(\lambda/(1 - q)) \exp_q(\epsilon P/(1 - q)) \). Thus, we obtain:

\[
\mathcal{J}_{\mathbb{C}P^1}^{\text{sym}}(\lambda + \epsilon P) = \mod (1-P)^2
\]

\[
(1 - q) \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{d=0}^{\infty} \frac{\lambda^m \epsilon^l P^d Q^d \prod_{r=0}^{d-1} (1 - \epsilon P q^r) \prod_{s=1}^{d} (1 - q^r) \prod_{r=1}^{d} (1 - P q^r)}{\prod_{t=1}^{m} (1 - q^t) \prod_{s=1}^{l} (1 - q^s) \prod_{r=1}^{d} (1 - P q^r)}.
\]

**Reconstruction theorems**

As in Example 1, assume that \( p_1, \ldots, p_K \) is a numerically effective integer basis in \( H^2(X, \mathbb{Q}) \), that Novikov’s monomials \( Q^d = Q_{1}^{d_1} \cdots Q_{K}^{d_K} \) represent degree \( d \) holomorphic curves in \( X \) in coordinates \( d_i = p_i(d) \) on \( H_2(X) \), that \( P_i \) are line bundles with \( \text{ch} P_i = e^{-p_i} \), and that monomials \( P^a = P_{1}^{a_1} \cdots P_{K}^{a_K} \) run a basis in \( K^0(X)_{\text{pr}} \), the primitive part of the K-ring. We also write \( a.d \) for the value \( \sum a_i d_i \) of \( -c_1(P^a) \) on \( d \).
**Theorem 2** (explicit reconstruction). Let $I = \sum_d I_d Q^d$ be a point in the range $L \subset K$ of the $J$-function of permutation-equivariant quantum $K$-theory on $X$, written as a vector-valued series in Novikov’s variables. Then the following family also lies in $L$:

$$\sum_d I_d Q^d \epsilon \sum_{k>0} \sum_a \psi^k(\epsilon_a) P^{ka} q^{k(a,d)}/k(1-q^k), \quad \epsilon_a \in \Lambda_+.$$ 

Moreover, for arbitrary Laurent polynomials $c_a \in \Lambda[q,q^{-1}]$, the following series also lies in $L$:

$$\sum_d I_d Q^d \epsilon \sum_{k>0} \sum_a \psi^k(\epsilon_a) P^{ka} q^{k(a,d)}/k(1-q^k) \sum_a c_a(q) P^a q^{a.d}.$$ 

Furthermore, when $K^0(X) = K^0(X)_{pr}$, the whole cone $L \subset K$ is parameterized this way.

**Proof.** We first work over $\Lambda'$ freely generated as $\lambda$-algebra by the “time” variables $\epsilon_a$, and use Theorem 1 with the $Q$-independent finite difference operator $D = \sum_a \epsilon_a P^a q^{aQd}$. We conclude that the family

$$\sum_{k>0} \sum_a \psi^k(\epsilon_a) P^{ka} q^{k(a,d)}/k(1-q^k) I = \sum_d I_d Q^d \epsilon \sum_{k>0} \sum_a \psi^k(\epsilon_a) P^{ka} q^{k(a,d)}/k(1-q^k)$$

lies in the cone $L$, defined over $\Lambda'$. To obtain the second statement, we apply Proposition, using finite difference operators $\sum_a c_a(q) P^a q^{aQd}$. Afterwards we specialize the “times” $\epsilon_a$ to any values $\epsilon_a \in \Lambda$ (which at this point may become dependent on $Q$). Finally, when $K^0(X) = K^0(X)_{pr}$, we use the formal Implicit Function Theorem to conclude that the whole cone $L$ is parameterized, because this is true modulo Novikov’s variables, as Example 1 shows. \qed

**Example:** $X = \mathbb{CP}^N$. According to Theorem 2, the entire cone $L$ is parameterized as follows:

$$J = (1-q) \sum_{d \geq 0} Q^d \epsilon \sum_{k \geq 0} \sum_{a=0}^N \psi^k(\epsilon_a) P^{ka} q^{k(a,d)}/k(1-q^k) \sum_{a=0}^N c_a(q) P^a q^{a.d}$$

$$\frac{1}{(1-Pq)^{N+1}(1-Pq^2)^{N+1} \cdots (1-Pq^d)^{N+1}}.$$ 

Of course, this is obtained by applying Theorem 2 to the small $J$-function $J(0)$ from [5] (also [6], or Parts II–IV in the non-equivariant limit). Here $\epsilon_a \in \Lambda$, $c_a(q)$ are arbitrary Laurent polynomials in $q$ with coefficients in $\Lambda$, and $P_a$, $a = 0, \ldots, N$, $P = \mathcal{O}(-1)$, are used for a basis in $K^0(X)$. Perhaps, the basis $(1-P)^a, a = 0, \ldots, N$, is more useful (cf. [4]), and we get yet another parameterization of $L$:

$$(1-q) \sum_{d \geq 0} Q^d \epsilon \sum_{k \geq 0} \sum_{a=0}^N \psi^k(\epsilon_a) (1-P^k q^{k(a,d)}/k(1-q^k) \sum_{a=0}^N c_a(q) (1-Pq^a)^d)$$

$$\frac{1}{(1-Pq)^{N+1}(1-Pq^2)^{N+1} \cdots (1-Pq^d)^{N+1}}.$$
We return now to the context of Part VII, where we studied the mixed J-function \( J(x, t) \), involving two types of inputs: permutable \( t \) and non-permutable \( x \), both taken from \( K_+ \). The cone \( L \subset K \) represents the range of \( t \mapsto J(0, t) \). Recall that according to the general theory, it is the union of ruling spaces \( (1 - q)T_t \), where \( t = T(t) \) is given by a certain non-linear map 

\[
T : K^0(X) \otimes \Lambda_+ \oplus (1 - q)K_+ \rightarrow K^0(X) \otimes \Lambda_+.
\]

At the same time, for a fixed value of \( t \), the range of the ordinary J-function \( x \mapsto J(x, t) \) is an overruled Lagrangian cone \( L_t \subset K \), which shares with \( L \) one ruling space, \( T_t \), corresponding to \( t = T(t) \). Each tangent space of each cone \( L_t \) is tangent to \( L_t \) along one of the ruling spaces (e.g. \( T_t \) is tangent along \( (1 - q)T_t \)), and is related with this ruling space by the multiplication by \( 1 - q \). As a consequence, not only each ruling (and tangent) space of each \( L_t \) is a \( D_q \)-module (which is proved on the basis of adelic characterization as in Proposition above), but also each cone \( L_t \) is invariant under the flow 

\[
f \mapsto e^{eD(Q, P_q^{Q^2, Q})/(1-q)} f,
\]

where \( D \in D_q \). We use this to reconstruct the family \( L_t \).

**Theorem 3.** Let \( I = \sum I_d Q^d \) (as in Theorem 2). Then

\[
I(\epsilon) = \sum_d I_d(\epsilon) Q^d := \sum_d I_d(\epsilon)Q^d e^{\sum k > 0 \sum a \Psi^k(\epsilon_a) P^{a \cdot \Psi(k \cdot a \cdot d)}/k(1-q^k)}, \quad \epsilon_a \in \Lambda_+
\]

represent a family of points on the cones \( L_{t(\epsilon)} \) (one point on each cone), and the following family of points, parameterized by \( \tau_a \in \Lambda \) and by \( c_a \in \Lambda[q, q^{-1}] \), lies on \( L_{t(\epsilon)} \):

\[
\sum_d I_d(\epsilon) Q^d e^{\sum_a \tau_a P^a Q^a / (1-q)} \sum_a c_a(q) P^a q^{a \cdot d}.
\]

Moreover, if \( K^0(X) = K^0(X)_{pr} \), for each \( t \in K^0(X) \otimes \Lambda_+ \) the whole cone \( L_t \) is thus parameterized.

**Proof.** It is clear from computation modulo Novikov’s variables that the family \( I(\epsilon) \) has no tangency with the ruling spaces, hence represents at most one point from each \( L_t \) (and does represent one, when \( K^0(X) = K^0(X)_{pr} \)). Given one point, \( I(\epsilon) \), on \( L_{t(\epsilon)} \), we generate more points by machinery discussed above: applying the commuting flows 

\[
e^{\sum \tau_a P^a_q Q^a Q^a / (1-q)} I(\epsilon) = \sum_d Q^d I_d(\epsilon) e^{\sum \tau_a P^a_q q^{a \cdot d} / (1-q)},
\]
followed by the application of the operators $\sum a \, c_a(\tau) P^a q^a Q^{\partial_2}$, where $\tau_a$ and the coefficients of $c_a$ are independent variables. Afterwards they can be specialized to some values in $\Lambda$ (in particular, depending on $Q$). In the case when $K^0_0(X) = K^0_0(X)_{\text{pr}}$, it follows from the Implicit Function Theorem and Example 1 about the limit to $d = 0$, that the entire cone $\mathcal{L}_t$ for each $t$ is thus obtained.

**Example:** $X = \mathbb{C}P^N$. It follows that for fixed values of $\epsilon_a$, the corresponding cone $\mathcal{L}_{t(\epsilon)}$ is parameterized as

$$(1 - \tau) \sum_{d \geq 0} Q^d \sum_{a=0}^N \left( \sum_{k > 0} \Psi^k(\epsilon_a) P^a q^{k+d} \right) \sum_{a=0}^N c_a(\tau) P^a q^{kad}$$

and all $\mathcal{L}_t$ are so obtained.

**Remarks.** Reconstruction theorems in quantum cohomology and K-theory go back to Kontsevich–Manin [9] and Lee–Pandharipande [10] respectively. Theorem 3 is a slight generalization (from the case $t = 0$) of the “explicit reconstruction” result [4] in the ordinary (non-permutation-equivariant) quantum K-theory, which in its turn mimics the results of quantum cohomology theory already found in [1, 7], and shares the methods based on finite difference operators with the K-theoretic results of [8].

Theorems of this section show that when $K^0_0(X)$ is generated by line bundles, the entire range $\mathcal{L}$ of the J-function in the permutation-equivariant genus-0 quantum K-theory of $X$, as well as the entire family $\mathcal{L}_t$ of the overruled Lagrangian cones representing the “ordinary” J-functions, depending on the permutable parameter, $t$, can be explicitly represented in a parametric form, **given one point on any of these cones**. In essence, all genus-0 K-theoretic GW-invariants of $X$, permutation-equivariant, ordinary, or mixed, are thereby reconstructed from any one point: a $K^0_0(X)$-valued series $\sum I_d Q^d$ in Novikov’s variables.

In the case of a toric $X$, the results of Part V exhibit such a point in the form of the $q$-**hypergeometric series** mirror-symmetric to $X$. Needless to say, the same applies to toric bundles spaces, or super-bundles (a.k.a. toric complete intersections), as well as to the torus-equivariant versions of K-theoretic GW-invariants. Thus “all” (torus-equivariant or not; permutation-equivariant, ordinary, or mixed) K-theoretic genus-0 GW-invariants of toric manifolds, toric bundles, or toric complete intersections are computed in a geometrically explicit form, illustrated by the above example.
References