PERMUTATION-EQUIVARIANT QUANTUM K-THEORY VII. GENERAL THEORY

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Abstract. We introduce K-theoretic GW-invariants of mixed nature: permutation-equivariant in some of the inputs and ordinary in the others, and prove the ancestor-descendant correspondence formula. In genus 0, combining this with adelic characterization, we derive that the range $\mathcal{L}_X$ of the big J-function in permutation-equivariant theory is overruled.

The string and dilaton equations

We return to the introductory setup of Part I, and introduce *mixed genus-g descendant potentials* of a compact Kähler manifold X:

$$\mathcal{F}_g(\mathbf{x},\mathbf{t}) := \sum_{k\geq 0, n\geq 0, d} \frac{Q^d}{k!} \langle \mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle^{S_n}_{g,k+n,d}.$$

The first k seats are occupied by the input $\mathbf{x} = \sum_{r\in\mathbb{Z}} x_r q^r$, which is a Laurent polynomial in q with vector coefficients $x_r \in K^0(X)\otimes\Lambda$. We assume that Λ includes Novikov's variables as well. The last n seats are occupied by similar inputs $\mathbf{t} = \sum_{r\in\mathbb{Z}} \mathbf{t}_r q^r$, $\mathbf{t}_r \in K^0(X)\otimes\Lambda$, and only these inputs are considered *permutable* by renumberings of the marked points. Most of the time we will assume that $\mathbf{t}(q) = t$ is constant in q, i.e. that the permutable inputs do not involve the cotangent line bundles L_i.

We will first treat these generating functions as objects of the *ordinary*, i.e. permutation-*non*-equivariant, quantum K-theory, depending however on the parameter t. Our nearest aim is to extend to this family of theories some basic facts from the ordinary GW-theory, starting with the genus-0 string and dilaton equations.

On the moduli space $X_{g,m+1,d}$, along with the line bundles L_i formed by the cotangent lines to the curves at the ith marked point, consider

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the line bundles $\widetilde{L}_i := \mathrm{ft}_1^*(L_i)$, $i \geq 2$, where $\mathrm{ft}_1 : X_{g,1+m,d} \to X_{g,m,d}$ is the map defined by forgetting the first marked point. In the *genus 0* case, it is clear that

$$\langle 1, \mathbf{x}_1(\widetilde{L}), \dots, \mathbf{x}_k(\widetilde{L}); t, \dots\rangle^{S_n}_{0,1+k+n,d} = \langle \mathbf{x}_1(L), \dots, \mathbf{x}_k(L); t, \dots\rangle^{S_n}_{0,k+n,d}.$$

On the other hand, it is well-known how to compare L_i and $\widetilde{L}_i$. The fibers of these line bundles coincide everywhere outside the section $\sigma_i : X_{g,m,d} \to X_{g,1+m,d}$ defined by the ith marked point, while $\sigma_i^*(\widetilde{L}_i) = L_i$, and $\sigma_i^*(L_i) = 1$. In other words, $1 - \widetilde{L}_i/L_i = (\sigma_i)_* 1$, and hence $\widetilde{L}_i = L_i - (\sigma_i)_* 1$. Taking into account that $L_i((\sigma_i)_* 1) = (\sigma_i)_* 1$, and that $((-\sigma_i)_* 1)^r = (-\sigma_i)_*(L_i - 1)^{r-1}$, we find by Taylor's formula (and omitting the subscript i):

$$\mathbf{x}(\widetilde{L}) - \mathbf{x}(L) = \sum_{r>0} \frac{\mathbf{x}^{(r)}(L)}{r!}(-\sigma_* 1)^r =$$

$$-\sigma_* \left(\sum_{r>0} \frac{\mathbf{x}^{(r)}(1)}{r!}(L-1)^{r-1} \right) = -\sigma_* \frac{\mathbf{x}(L) - \mathbf{x}(1)}{L-1}.$$

Note that the divisors σ_i for different i are disjoint, and that $\sigma_i^*(L_j) = L_j$ if $j \neq i$. Thus

$$\langle 1, \mathbf{x}_1(L), \dots, \mathbf{x}_k(L); t, \dots\rangle^{S_n}_{0,1+k+n,d} = \langle \mathbf{x}_1(L), \dots, \mathbf{x}_k(L); t, \dots\rangle_{0,k+n,d} +$$

$$\sum_{i=1}^k \langle \dots, \mathbf{x}_{i-1}(L), \frac{\mathbf{x}_i(L) - \mathbf{x}_i(1)}{L-1}, \mathbf{x}_{i+1}(L), \dots; t, \dots, t\rangle^{S_n}_{0,k+n,d}.$$

This computation is quite standard, since it does not interfere with the permutable inputs, as long as those don't contain line bundles L_i.

`Proposition 1` (string equation). *Let V be the linear vector field on the space of vector-valued Laurent polynomials in q defined by*

$$V(\mathbf{y}) := \frac{\mathbf{y}(q) - \mathbf{y}(1)}{1-q}.$$

In the genus-0 descendent potential $F_0(\mathbf{x}, t)$, introduce the dilaton shift *of the origin: $\mathbf{y}(q) = 1 - q + t + \mathbf{x}(q)$. Then*

$$L_V(\mathcal{F}_0(\mathbf{y}+q-1-t), t)) = \mathcal{F}_0(\mathbf{y}+q-1-t), t) + \frac{(\mathbf{y}(1), \mathbf{y}(1))}{2} - \left(\frac{\Psi^2(t)}{2}, 1\right),$$

where $(a, b) := \chi(X; ab)$ is the Λ-valued K-theoretic Poincaré pairing, and Ψ^2 is the 2nd Adams operation on $K^0(X) \otimes \Lambda$.

`Proof.` The linear vector field V becomes inhomogeneous in the unshifted coordinate system:

$$\frac{\mathbf{y}(q)-\mathbf{y}(1)}{1-q}=\frac{\mathbf{x}(q)-\mathbf{x}(1)}{1-q}+1.$$

Applying the previous, down-to-earth form of the string equation to

$$\mathcal{F}_0(\mathbf{x}):=\sum_{k,n,d}\frac{Q^d}{k!}\langle\mathbf{x}(L),\ldots,\mathbf{x}(L);t,\ldots,t\rangle_{0,1+k+n,d}^{S_n},$$

we gather that

$$L_V(\mathcal{F}_0(\mathbf{x},t))=\mathcal{F}_0(\mathbf{x})+\text{terms }\langle 1,\ldots\rangle_{0,3,0}^{S_n}\text{ with }d=0\text{ and }k+n=2.$$

Since $X_{0,3,0}=X\times\overline{\mathcal{M}}_{0,3}=X$, and $L=1$ on $\overline{\mathcal{M}}_{0,3}$, these terms are

$$\frac{1}{2}(\mathbf{x}(1),\mathbf{x}(1))+(\mathbf{x}(1),t)+\frac{1}{2}(t,t)/2-\frac{1}{2}(\Psi^2(t),1).$$

The last two terms come from

$$\langle 1;t,t\rangle_{0,3,0}^{S_2}=\frac{1}{|S_2|}\sum_{h\in S_2}\mathrm{tr}_h(t^{\otimes 2}).$$

All but the last one add up to $(y(1),y(1))/2$. $\square$

Consider now correlators

$$\langle L-1,\mathbf{x}(L),\ldots,\mathbf{x}(L);\mathbf{t}(L),\ldots,\mathbf{t}(L)\rangle_{0,1+k+n,d}^{S_n}.$$

The line bundle L_1 over $X_{g,1+k+n,d}$ differs from the dualizing sheaf to the fibers of the forgetting map $\mathrm{ft}_1 : X_{g,1+k+n,d}\to X_{g,k+n,d}$ by the divisor of the marked points. The spaces $H^0(\Sigma,L)$ are formed by holomorphic differentials on Σ with at most 1st order poles at the markings, and with at most 1st order poles at the nodes with zero residue sum at each node. In genus 0, if $k+n>0$, then $H^1(\Sigma,L-1)=0$, while the holomorphic differentials are uniquely determined by the residues at the marked points subject to the constrains that the total sum is 0. The residues *per se* form trivial bundles, but those at the permutable marked points form the standard Coxeter representation of S_n, induced from the trivial representation of S_{n-1}. Thus,

$$(ft_1)_*(L-1)=k-2+\mathrm{Ind}_{S_{n-1}}^{S_n}(1),$$

and this answer is correct even when $k=n=0$ (in which case $H^1(\Sigma,L)=H^0(\Sigma,1)^*=1$). On the other hand, $L-1$ vanishes on the sections $\sigma_i : X_{g,k+l,d} \to X_{g,1+k+n,d}$ defined by the markings, where the differences between $\widetilde{L}_i - L_i$, $i > 1$, are supported. We find that

$$\langle L-1,\dots,\mathbf{x}(L);\mathbf{t}(L),\dots\rangle^{S_n}_{0,1+k+n,d} = (k-2)\langle\dots,\mathbf{x}(L);\mathbf{t}(L),\dots\rangle^{S_n}_{0,k+n,d}$$
$$+\langle\mathbf{x}(L)\dots,\mathbf{x}(L),\mathbf{t}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle^{S_{n-1}}_{0,k+n,d}.$$

We use here that for any S_n-module V,

$$\left(V\otimes \mathrm{Ind}^{S_n}_{S_{n-1}}(1)\right)^{S_n} = \left(\mathrm{Res}^{S_n}_{S_{n-1}}(V)\right)^{S_{n-1}}.$$

`Proposition 2` (dilaton equation). *The genus-0 descendent potential $\mathcal{F}_0$ in dilaton-shifted coordinates satisfies the following homogeneity condition:*

$$L_E(\mathcal{F}_0(\mathbf{y}+q-1-\mathbf{t},\mathbf{t}) = 2\mathcal{F}_0(\mathbf{y}+q-1-\mathbf{t},\mathbf{t}) + (\Psi^2(\mathbf{t}(1)),1),$$

where E is the Euler vector field $E(\mathbf{y})=\mathbf{y}$ in the linear space of vector-valued Laurent polynomials $\mathbf{y}(q)$.

`Proof.` The exceptional terms

$$\frac{1}{2}\langle L-1,\mathbf{x}(L),\mathbf{x}(L)\rangle_{0,3,0}+\langle L-1,\mathbf{x}(L);\mathbf{t}(L)\rangle^{S_1}_{0,3,0}+\langle L-1;\mathbf{t}(L),\mathbf{t}(L)\rangle^{S_2}_{0,3,0}$$

all vanish except for the trace of the non-trivial element in S_2, which acts by -1 one the cotangent line L. This makes $L-1$ on $\overline{\mathcal{M}}_{0,3}$ equal to -2 (rather than 0), and results in the constant $-(\Psi^2(\mathbf{t}(1)),1)$. Therefore the identity derived above yields:

$$\sum_{k,n,d}\frac{Q^d}{k!}\langle 1-L+\mathbf{t}(L)+\mathbf{x}(L),\mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle^{S_n}_{0,1+k+n,d}$$
$$=2\sum_{k,n,d}\frac{Q^d}{k!}\langle\mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle^{S_n}_{0,k+n,d}+(\Psi^2(\mathbf{t}(1)),1),$$

which after shift $\mathbf{y}(q) := 1-q+\mathbf{t}(q)+\mathbf{x}(q)$ becomes what we claimed. □

`Remark.` Note that we have proved this allowing the permutable input $\mathbf{t}$, i.e. the *parameter* of $\mathcal{F}_0$ to depend on q.

A WDVV-equation

Let us introduce the gadget

$$\langle\langle A_1,\dots,A_m\rangle\rangle_{g,m} := \sum_{l,n,d}\frac{Q^d}{l!}\langle A_1,\dots,A_m;\tau,\dots,\tau;t,\dots,t\rangle^{S_n}_{g,m+l+n,d},$$

for the generating function of $\tau, t\in K^0(X)\otimes\Lambda$, and the meaning of the inputs A_i to be specified.

Along with the *Poincaré metric* $g_{\alpha\beta} = (\phi_\alpha, \phi_\beta)$ on $K^0(X)$, where $\{\phi_\alpha\}$ is a basis, introduce the non-constant metric

$$G_{\alpha\beta} := g_{\alpha\beta} + \langle\langle \phi_\alpha, \phi_\beta \rangle\rangle_{0,2}.$$

Note that the inverse tensor has the form

$$G^{\alpha\beta} = g^{\alpha\beta} - \langle\langle \phi^\alpha, \phi^\beta \rangle\rangle_{0,2} + \sum_\mu \langle\langle \phi^\alpha, \phi^\mu \rangle\rangle_{0,2} \langle\langle \phi_\mu, \phi^\beta \rangle\rangle_{0,2}$$
$$- \sum_{\mu,\nu} \langle\langle \phi^\alpha, \phi^\mu \rangle\rangle_{0,2} \langle\langle \phi_\mu, \phi^\nu \rangle\rangle_{0,2} \langle\langle \phi_\nu, \phi^\beta \rangle\rangle_{0,2} + \dots,$$

where $\{\phi^\alpha\}$ is the basis Poincar'e-dual to $\{\phi_\alpha\}$.

Proposition 3 (WDVV-equation). *For all $\phi, \psi \in K^0(X) \otimes \Lambda$,*

$$(\phi, \psi) + (1 - xy) \langle\langle \frac{\phi}{1 - xL}, \frac{\psi}{1 - yL} \rangle\rangle_{0,2} =$$
$$\sum_{\alpha,\beta} \left((\phi, \phi_\alpha) + \langle\langle \frac{\phi}{1 - xL}, \phi_\alpha \rangle\rangle_{0,2} \right) G^{\alpha\beta} \left((\phi_\beta, \psi) + \langle\langle \phi_\beta, \frac{\psi}{1 - yL} \rangle\rangle_{0,2} \right).$$

Proof. The standard WDVV-argument consists in mapping moduli spaces of genus-0 stable maps with 4+ marked points to the Deligne-Mumford space $\overline{\mathcal{M}}_{0,4}$, and considering the inverse image of a typical point, i.e., in other words, fixing the cross-ratio of the first 4 marked points. When the cross-ratio degenerates into one of the special values $0, 1, \infty$, the curves become reducible, with the 4 marked points split into pairs between the two glued pieces in 3 different ways. The WDVV-equation expresses the equality between the three gluings.

We apply the argument to the inputs of the 4 marked points equal to $1, 1, \phi/(1-xL)$, and $\phi/(1-yL)$, and arrive at the following identity (see Figure 1):

$$\sum_{\alpha,\beta} \langle\langle 1, \frac{\phi}{1 - xL}, \phi_\alpha \rangle\rangle_{0,3} G^{\alpha\beta} \langle\langle \phi_\beta, \frac{\psi}{1 - yL}, 1 \rangle\rangle_{0,3} =$$
$$\sum_{\alpha,\beta} \langle\langle 1, 1, \phi_\alpha \rangle\rangle_{0,3} G^{\alpha\beta} \langle\langle \phi_\beta, \frac{\phi}{1 - xL}, \frac{\psi}{1 - yL} \rangle\rangle_{0,3}.$$

As it is explained in [6], in K-theory the WDVV-argument encounters the following subtlety. The virtual divisor obtained by fixing the cross-ratio and passing to any of the three limits, has self-intersections, represented by curves with more than 2 components (as shown on Figure 1 in shaded areas). As a result, the structure sheaf of the divisor before the limit is identified with the alternated sum of the structure

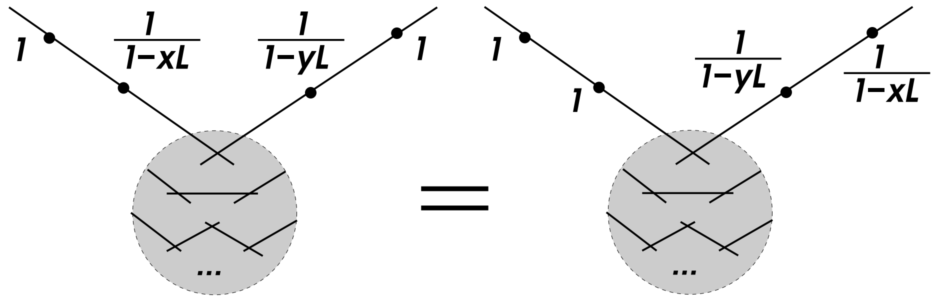

FIGURE 1. WDVV equation

sheaves of all the self-intersection strata on a manner of the exclusion-inclusion formula. In the identity, this is taken care of by the pairing which involves the tensor $G^{\alpha\beta}$.

It only remains to apply the string equation. Since

$$\frac{1}{L-1}\left(\frac{1}{1-qL}-\frac{1}{1-q}\right)=\frac{x}{(1-x)}\frac{1}{(1-qL)},$$

and since $L=1$ on $X_{0,3,0}=X\times\overline{\mathcal{M}}_{0,3}=X$, we have

$$\langle\langle 1,\frac{\phi}{1-qL},\phi_\alpha\rangle\rangle_{0,3}=\frac{(\phi,\phi_\alpha)}{1-q}+\left(1+\frac{q}{1-q}\right)\langle\langle\frac{\phi}{1-qL},\phi_\alpha\rangle\rangle_{0,2};$$

$$\sum_{\alpha,\beta}\langle\langle 1,1,\phi_\alpha\rangle\rangle_{0,3}G^{\alpha\beta}\langle\langle\phi_\beta,\frac{\phi}{1-xL},\frac{\psi}{1-yL}\rangle\rangle_{0,3}=\langle\langle 1,\frac{\phi}{1-xL},\frac{\psi}{1-yL}\rangle\rangle_{0,3}$$

$$=\frac{(\phi,\psi)}{(1-x)(1-y)}+\left(1+\frac{x}{1-x}+\frac{y}{1-y}\right)\langle\langle\frac{\phi}{1-xL},\frac{\psi}{1-yL}\rangle\rangle_{0,2}.$$

The result follows. □

THE LOOP SPACE FORMALISM

Here we interpret the string, dilaton, and WDVV-equations using symplectic linear algebra in the space $\mathcal{K}$ of rational functions of q with vector coefficients from $K^0(X)\otimes\Lambda$. To be more precise, we assume that elements of $\mathcal{K}$ are such rational functions *modulo any power of Novikov's variables* (or in any other topology that may turn out useful in future). We equip $\mathcal{K}$ with symplectic form

$$\Omega(\mathbf{f},\mathbf{g}):=-\operatorname{Res}_{q=0,\infty}(\mathbf{f}(q^{-1}),\mathbf{g}(q))\,\frac{dq}{q}.$$

We identify $\mathcal{K}$ with $T^*\mathcal{K}_+$ where $\mathcal{K}_+\subset\mathcal{K}$ is the Lagrangian subspace consisting of vector-valued Laurent polynomials in q (in the aforementioned topological sense) by picking the complementary Lagrangian subspace $\mathcal{K}_-$ consisting of rational functions of q regular at $q=1$ and vanishing at $q=\infty$. We encode K-theoretic genus-0 GW-invariant of X by the *big J-function*

$$\mathcal{J}(\mathbf{x},\mathbf{t}) := 1-q+\mathbf{t}(q)+\mathbf{x}(q)+ \sum_{\alpha,k,n,d} \phi^\alpha \frac{Q^d}{k!}\langle \frac{\phi_\alpha}{1-qL}, \mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle_{0,1+k+n,d}^{S_n}.$$

`Proposition 4.` *The big J-function is the dilaton-shifted graph of the differential of the genus-0 descendent potential $\mathcal{F}_0$:*

$$\mathcal{J}(\mathbf{x},\mathbf{t}) = 1-q+\mathbf{t}+\mathbf{x}+d_{\mathbf{x}}\mathcal{F}_0(\mathbf{x},\mathbf{t}).$$

`Proof.` For every $\mathbf{v}\in\mathcal{K}_+$, we have:

$$L_{\mathbf{v}}\mathcal{F}_0 = \sum_{k,n,d}\frac{Q^d}{k!}\langle \mathbf{v}(L),\mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle_{0,1+k+n,d}^{S_n} =$$

$$\sum_{k,n,d}\frac{Q^d}{k!}\langle \mathrm{Res}_{q=L}\,\frac{\mathbf{v}(q)}{(1-L/q)}\,\frac{dq}{q},\mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle_{0,1+k+n,d}^{S_n}$$

$$= -\,\mathrm{Res}_{q=0,\infty}(\mathcal{J}(q^{-1}),\mathbf{v}(q))\,\frac{dq}{q} = \Omega(\mathcal{J},\mathbf{v}).$$

`Corollary 1` (dilaton equation). *For a fixed value of the parameter* $\mathbf{t}$*, the range of the J-function* $\mathbf{x}\mapsto\mathcal{J}(\mathbf{x},\mathbf{t})$ *is a Lagrangian cone* $\mathcal{L}_{\mathbf{t}}\subset\mathcal{K}$ *with the vertex at the origin.*

`Proof.` Differentiating the dilaton equation for $\mathcal{F}_0$, we find that 1st derivatives of $\mathcal{F}_0$ are homogeneous of degree 1. $\square$

`Corollary 2` (string equation). *For any* $t\in K^0(X)\otimes\Lambda$*, the linear vector field* $\mathbf{f}\mapsto\mathbf{f}/(1-q)$ *on* $\mathcal{K}$ *is tangent to* $\mathcal{L}_t$*.*

`Remark.` We will see later that this is true for any $\mathcal{L}_{\mathbf{t}}$, and not only for $\mathbf{t}$ independent of q.

`Proof.` Subtracting from the string equation for $\mathcal{F}_0$ derived in the previous section a half of the dilaton equation for $\mathcal{F}_0$, we obtain a Hamilton-Jacobi equation $L_{V-E/2}\mathcal{F}_0 = (\mathbf{y}(1),\mathbf{y}(1))/2$. It expresses the fact that the quadratic Hamiltonian corresponding to this equation vanishes on $\mathcal{L}_t$, and hence the Hamiltonian vector field is tangent to $\mathcal{L}_t$. We will show that this Hamiltonian vector field is

$$W\mathbf{f} := \frac{\mathbf{f}}{1-q}-\frac{\mathbf{f}}{2}.$$

Due to Corollary 1, $\mathbf{f}\mapsto\mathbf{f}/2$ is tangent to $\mathcal{L}_t$, and the result about $\mathbf{f}\mapsto\mathbf{f}/(1-q)$ would follow.

The hamiltonian of W is $H(\mathbf{f}) := \Omega(\mathbf{f}, W\mathbf{f})/2 = \Omega(\mathbf{f}, \mathbf{f}/(1-q))/2$. Using the projections $\mathbf{f}_\pm$ of $\mathbf{f} \in \mathcal{K}$ to $\mathcal{K}_\pm$, we compute $2H(\mathbf{f})$:

$$\Omega\left(\mathbf{f}, \frac{\mathbf{f}}{1-q}\right) = \Omega\left(\mathbf{f}_+ + \mathbf{f}_-, \frac{\mathbf{f}_+(1)}{1-q} + \frac{\mathbf{f}_+ - \mathbf{f}_+(1)}{1-q} + \frac{\mathbf{f}_-}{1-q}\right) =$$

$$-\Omega\left(\frac{\mathbf{f}_+(1)}{1-q}, \mathbf{f}_+\right) + \Omega\left(\mathbf{f}_-, \frac{\mathbf{f}_+ - \mathbf{f}_+(1)}{1-q}\right) + \Omega\left(\frac{\mathbf{f}_+}{1-q^{-1}}, \mathbf{f}_-\right) =$$

$$\operatorname{Res}_{q=0,\infty}\left(\frac{\mathbf{f}_+(1)}{1-q^{-1}}, \mathbf{f}_+(q)\right)\frac{dq}{q} + \Omega\left(\mathbf{f}_-, \frac{\mathbf{f}_+ - 2\mathbf{f}_+(1) + q\mathbf{f}_+}{1-q}\right) +$$

$$\Omega\left(\mathbf{f}_-, \frac{\mathbf{f}_+(1)}{1-q}\right) = -(\mathbf{f}_+(1), \mathbf{f}_+(1)) + 2\,\Omega\left(\mathbf{f}_-, \frac{\mathbf{f}_+ - \mathbf{f}_+(1)}{1-q} - \frac{\mathbf{f}_+}{2}\right) + 0.$$

The last non-zero term is twice the Hamilton function of the vector field $\mathbf{y} \mapsto \frac{\mathbf{y}(q)-\mathbf{y}(1)}{1-q} - \mathbf{y}(q)/2$ on $\mathcal{K}_+$, i.e. $V - E/2$, lifted to the cotangent bundle in the standard way. The first non-zero term is twice $-(\mathbf{y}(1), \mathbf{y}(1))/2$. Thus, the quadratic hamiltonian H is exactly as claimed. $\square$

Introduce the operator $S : K^0(X) \otimes \Lambda \to \mathcal{K}_-$ defined by

$$S(q)\,\phi = \sum_{\alpha,\beta}\left((\phi, \phi_\alpha) + \langle\langle \frac{\phi}{1-L/q}, \phi_\alpha \rangle\rangle_{0,2}\right) G^{\alpha\beta}\phi_\beta,$$

The operator depends on the parameter $\tau \in K^0(X) \otimes \Lambda$. For each value of the parameter, it can be considered as an operator-valued rational function of q (a "loop group" element), and in this capacity extends to a map $S : \mathcal{K} \to \mathcal{K}$ commuting with multiplications by scalar rational functions of q. The WDVV-identity from the previous section can be written as

$$(1-xy)\langle\langle \frac{\phi}{1-xL}, \frac{\psi}{1-yL} \rangle\rangle_{0,2} = (\phi, \psi) + \left(S^*(y^{-1})S(x^{-1})\phi, \psi\right),$$

where

$$S^*(q)\psi = \psi + \sum_{\alpha\beta}\langle\langle \psi, \frac{\phi_\alpha}{1-L/q} \rangle\rangle_{0,2} g^{\alpha\beta}\phi_\beta$$

is the operator adjoint to $S(q)$ with respect to the inner product $(g_{\alpha\beta})$ on the domain space, and $(G_{\alpha\beta})$ on the target space. It follows that

$$S^*(q^{-1})S(q) = 1, \quad \text{and hence} \quad S(q)S^*(q^{-1}) = 1.$$

This means that $S : (\mathcal{K}, \Omega) \to (\mathcal{K}, \bar{\Omega})$ provides a symplectic isomorphism between two symplectic structures on the loop space: Ω, based on the metric tensor $(g_{\alpha\beta})$, and $\bar{\Omega}$, based on the metric tensor $(G_{\alpha\beta})$

(and depending therefore on the parameter $\tau \in K^0(X) \otimes \Lambda$). The inverse isomorphism is given by

$$S^{-1}(q) = S^*(q^{-1}).$$

Furthermore, the *quantizations* $\widehat{S}$ and $\widehat{S^{-1}}$ provide isomophisms between the corresponding Fock spaces, which in their formal version consist of expressions

$$\mathcal{D}(\mathbf{y}) = e^{\mathcal{F}_0(\mathbf{y})/\hbar + \mathcal{F}_1(\mathbf{y}) + \hbar\mathcal{F}_2(\mathbf{y}) + \hbar^2\mathcal{F}_3(\mathbf{y}) + \cdots}, \quad \mathbf{y} \in \mathcal{K}_+,$$

where $\mathcal{F}_g$ is a sequence of scalar-valued functions on $\mathcal{K}_+$.

Proposition 5. *The action of the quantized operator S^{-1} on an element $\mathcal{A}$ of the Fock space, corresponding to the symplectic form $\bar{\Omega}$, is given by*

$$(\widehat{S^{-1}}\mathcal{A})(\mathbf{y}) = e^{\langle\langle \mathbf{y}(L), \mathbf{y}(L)\rangle\rangle_{0,2}/\hbar}\mathcal{A}([S(q)\mathbf{y}(q)]_+),$$

where $[f(q)]_+$ denotes taking the Laurent polynomial part of rational function $\mathbf{f}$, i.e. the projection $\mathcal{K} \to \mathcal{K}_+$ along $\mathcal{K}_-$.

Proof. Generally speaking, quantization of linear symplectic transformations T is defined as $\exp \widehat{\ln T}$, where $\widehat{\ln T}$ is quantization of the quadratic hamiltonian according to the standard rules [7]. Namely, in Darboux coordinates on $\mathcal{K} = T^*\mathcal{K}_+$,

$$\widehat{q_\alpha q_\beta} = \hbar^{-1} q_\alpha q_\beta, \ \widehat{q_\alpha p_\beta} = q_\alpha \partial_{q_\beta}, \ \widehat{p_\alpha p_\beta} = \hbar \partial_{p_\alpha}\partial_{p_\beta}.$$

The operators $S^{\pm 1}$ have the form of the composition of the operator $G^{\pm 1}$ identifying the metric: $(G\phi_\mu, \phi_\nu) = G_{\mu\nu}$, and the operator $(\mathcal{K}, \Omega) \to (\mathcal{K}, \Omega)$ which is the identity modulo $\mathcal{K}_-$. This means that the quadratic hamiltonian of $\ln S^{-1}G$ contains only pq-terms and q^2-terms, but no p^2-terms. Therefore $\widehat{S^{-1}} := \exp(\widehat{\ln S^{-1}G})G^{-1}$ will act by a linear change of variables followed by the multiplication by a quadratic form, both depending on S. The answer in the finite form is given by Proposition 5.3 in [7]:

$$(\widehat{S^{-1}}\mathcal{A})(\mathbf{y}) = e^{W(\mathbf{y},\mathbf{y})/\hbar}\mathcal{A}([S(q)\mathbf{y}(q)]_+),$$

where the symmetric bilinear form W is determined by

$$W(\mathbf{x}, \mathbf{y}) = (\Omega \otimes \Omega)\left(\frac{S^*(x^{-1})S(y^{-1}) - 1}{1 - xy}, \mathbf{x}(x) \otimes \mathbf{y}(y)\right).$$

Using the WDVV-equation, we find

$$\begin{aligned} W(\mathbf{x}, \mathbf{y}) =& \operatorname{Res}_{x=0,\infty} \operatorname{Res}_{y=0,\infty} \langle\langle \frac{\mathbf{x}(x)}{1 - L/x}, \frac{\mathbf{y}(y)}{1 - L/y} \rangle\rangle_{0,2} \frac{dx}{x}\frac{dy}{y} \\ =& \langle\langle \mathbf{x}(L), \mathbf{y}(L)\rangle\rangle_{0,2}. \end{aligned}$$

Ancestor – descendent correspondence

We introduce *ancestor potentials*

$$\bar{\mathcal{F}}_g(\mathbf{x},\tau,t) := \sum_{k\geq 0,d} \frac{1}{k!}\langle\langle \mathbf{x}(\bar{L},\dots,\mathbf{x}(\bar{L})\rangle\rangle_{g,k} =$$

$$\sum_{k,l,n,d} \frac{Q^d}{k!l!}\langle \mathbf{x}(\bar{L}),\dots,\mathbf{x}(\bar{L});\tau,\dots,\tau;t,\dots,t\rangle^{S_n}_{g,k+l+n,d},$$

where $\tau, t \in K^0(X)\otimes\Lambda$, and $\bar{L}$ in the ith position of the correlator represents the line bundle $\bar{L}_i$ over the moduli space $X_{g,k+l+n,d}$ of stable maps to X, obtained by pulling back the universal cotangent line bundle at the ith marked point over the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ by the *contraction* map $\mathrm{ct}: X_{g,k+l+n} \to \overline{\mathcal{M}}_{g,k}$. The latter is defined by forgetting the map to X and the last $k+n$ marked points, and contracting those components of the curve which have become unstable. We follow the exposition in Appendix 2 of [3] to relate descendent and ancestor potentials. The geometry of this relationship goes back to the paper of Kontsevich-Manin [10] and Getzler [4].

Let L be one of the cotangent line bundles over $X_{g,k+l+n,d}$ (say, the 1st one), and $\bar{L}$ its counterpart pulled back from $\overline{\mathcal{M}}_{g,k}$. They are the same outside the locus where the 1st marked point lies on a component to be contracted. This shows that there is a holomorphic section of $Hom(\bar{L},L)$ vanishing on the virtual divisor $j: D\to X_{g,n,d}$ formed by gluing genus g stable maps, carrying all but the 1st out of the first k marked points, with genus 0 stable maps, carrying the 1st one. In fact, like in the case of the of WDVV-equation, the divisor has self-intersections (see Figure 2), and we have to refer once again to [6] for a detailed discussion of the K-theoretic exclusion-inclusion formula

$$\mathcal{O}-\mathcal{O}(-D) = j_*\mathcal{O}_D - j_*\mathcal{O}_{D_{(2)}} + j_*\mathcal{O}_{D_{(3)}} - \cdots$$

which expresses $1-\bar{L}/L = \mathcal{O}-\mathcal{O}(-D)$ in terms of structure sheaves of the strata $D_{(m)}$ of m-tuple self-intersections.

We will use this relationship to rid systematically of L's in favor of $\bar{L}$'s in the correlators. For this, we will have to consider "mixed" correlators, which allow both L and $\bar{L}$ at the same seat. Let us use the notation $\langle \phi L^a\bar{L}^b|$ in correlator expressions which have the specified inputs (here $\phi\in K^0(X)\otimes\Lambda$) in the singled out (first) seat, provided that all other inputs in all terms of the expression are the same. For

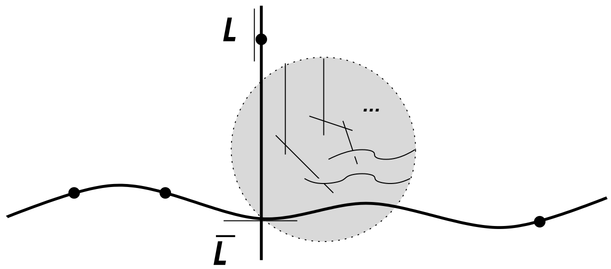

FIGURE 2. The divisor D and its self-intersections

$a > 0$, we have:

$$\begin{aligned}\langle \phi L^a \bar{L}^b| =& \langle \phi L^{a-1}\bar{L}^{b+1}| + \langle \phi L^a \bar{L}^b (1-\bar{L}/L)| \\ =& \langle \phi L^{a-1}\bar{L}^{b+1}| + \sum_{\alpha,\beta} \langle\!\langle \phi L^a, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^b| \ .\end{aligned}$$

Note that marked points (not shown on Figure 2) which carry the inputs τ or permutable inputs t can be distributed in any way between the components of the curve, and the above factorization of correlators under gluing is justified by the *permutation-equivariant binomial formula* from Part I.

Iterating the procedure, we have:

$$\begin{aligned}\langle \phi L^a| = \sum_{\alpha,\beta} \langle\!\langle \phi L^a, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta| + \cdots + \sum_{\alpha,\beta} \langle\!\langle \phi L, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^{a-1}| \\ + \langle \phi \bar{L}^a| = \langle \phi \bar{L}^a| + \sum_{b=0}^{a-1} \sum_{\alpha,\beta} \langle\!\langle \phi L^{a-b}, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^b|.\end{aligned}$$

Similarly, for negative exponents, we have

$$\begin{aligned}\langle \phi L^{-a-1}| &= \langle \phi L^{-a}\bar{L}^{-1}| - \langle (1-\bar{L}/L) L^{-1}/\bar{L}| \\ &= \langle \phi L^{-a}\bar{L}^{-1}| - \sum_{\alpha,\beta} \langle\!\langle \phi L^{-a}, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^{-1}| \\ &= \langle \phi L^{-a+1}\bar{L}^{-2}| - \cdots \\ &= \langle \phi \bar{L}^{-a-1}| - \sum_{b=0}^{a} \sum_{\alpha,\beta} \langle\!\langle \phi L^{-a+b}, \phi_\alpha \rangle\!\rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^{-b-1}|.\end{aligned}$$

In fact the result can be concisely described as

$$\langle \mathbf{x}(L)| = \Big\langle \big[S(\bar{L})\, \mathbf{x}(\bar{L}) \big]_+ \Big|,$$

where the operator S is as in the previous section:

$$S(q)\,\phi = \sum_{\alpha,\beta}\left((\phi,\phi_\alpha) + \langle\!\langle \frac{\phi}{1-L/q}, \phi_\alpha\rangle\!\rangle_{0,2}\right) G^{\alpha\beta}\phi_\beta,$$

and $[f(q)]_+$ means extracting from a rational function of q the Laurent polynomial part. The latter procedure, understood as projection along the space of rational functions regular at $q=0$ and vanishing at $q=\infty$, can be described by Cauchy's residue formula:

$$[f(q)]_+ = -\operatorname{Res}_{w=0,\infty} \frac{f(w)dw}{w-q}.$$

For a Laurent polynomial $\mathbf{x}(q)$ we have:

$$-\operatorname{Res}_{w=0,\infty} \frac{\mathbf{x}(w)}{(1-L/w)}\,\frac{dw}{(w-\bar{L})} = \frac{\mathbf{x}(L)}{1-\bar{L}/L} - \frac{\mathbf{x}(\bar{L})}{1-L/\bar{L}}.$$

For $\mathbf{x}(L) = L^a$ we get $(L^{a+1}-\bar{L}^{a+1})/(L-\bar{L}) = \sum_{b=0}^{a} L^{a-b}\bar{L}^b$, and hence

$$\left\langle \left[S(L)\,\phi \bar{L}^a\right]_+\right| =$$

$$\sum_{\alpha,\beta}\left((\phi,\phi_\alpha)G^{\alpha\beta}\langle\phi_\beta\bar{L}^a| + \sum_{b=0}^{a}\langle\!\langle \phi L^b, \phi_\beta\rangle\!\rangle_{0,2} G^{\alpha\beta}\langle\phi_\beta\bar{L}^{a-b}|\right),$$

which agrees with what we found earlier, because

$$\sum_\alpha (\phi,\phi_\alpha)G^{\alpha\beta} = (\phi,\phi_\beta) - \sum_\alpha \langle\!\langle\phi,\phi_\alpha\rangle\!\rangle_{0,2} G^{\alpha\beta}.$$

For $x(L) = L^{-a-1}$, it works out similarly:

$$\frac{L^{-a-1}}{1-\bar{L}/L} + \frac{\bar{L}^{-a-1})}{1-L/\bar{L}} == \frac{L^{-a}-\bar{L}^{-a}}{\bar{L}^{-1}-L^{-1}} = -\sum_{b=1}^{a} L^{-a+b}\bar{L}^{-b+1}.$$

The same procedure can be applied to each seat in the correlators. We conclude that for stable values of (g,m),

$\langle\!\langle \mathbf{x}(L),\dots,\mathbf{x}(L)\rangle\!\rangle_{g,m} = \langle\!\langle \mathbf{y}(\bar{L}),\dots,\mathbf{y}(\bar{L})\rangle\!\rangle_{g,m}$, where $\mathbf{y}(q) = [S(q)\mathbf{x}(q)]_+$

quite analogously to the cohomological results of [10]. Here all the correlators, as well as S, depend on the permutable parameters t and non-permutable τ. For a fixed t, assembling the correlators into the generating functions $\mathcal{F}_g$ and $\bar{\mathcal{F}}_g$, we find:

$$\mathcal{F}_g(\tau+\mathbf{x}) = \bar{\mathcal{F}}_g^{(\tau)}([S_\tau\mathbf{x}]_+) + \delta_{g,1}\langle\!\langle\ \rangle\!\rangle_{1,0}^{(\tau)} + \delta_{g,0}\sum_{m=0}^{2}\frac{1}{m!}\langle\!\langle \dots,\mathbf{x}(L),\dots\rangle\!\rangle_{0,m}^{(\tau)},$$

where the decorations by τ remind on the dependence on the parameter, and the terms on the right represent correlators with unstable values of $(g,m)=(1,0),(0,0),(0,1),(0,2)$, present in descendent, but absent in ancestor potentials.

Now we engage the shift of the origin $\mathbf{x}=\mathbf{y}+q-1-t-\tau$. We have:

$$\left[\frac{q-1-t-\tau}{1-L/q}\right]_+ = -\operatorname{Res}_{q=0,\infty}\frac{w-1-t-\tau}{1-L/w}\frac{dw}{w-q} = \\ \frac{q-1-t-\tau}{1-L/q}+\frac{L-1-t-\tau}{1-q/L} = L+q-1-t-\tau.$$

Therefore $[S(q)(q-1-t-\tau)]_+ =$

$$\sum_{\alpha,\beta}\left((q-1-t-\tau,\phi_\alpha)+\langle\!\langle\left[\frac{q-1-t-\tau}{1-L/q}\right]_+,\phi_\alpha\rangle\!\rangle_{0,2}\right)G^{|\alpha\beta}\phi_\beta \\ = q-1-t-\tau+\sum_{\alpha,\beta}\langle\!\langle L,\phi_\alpha\rangle\!\rangle_{0,2}G^{\alpha\beta}\phi_\beta = q-1.$$

The last equality is due to the string and dilaton equations:

$$\langle\!\langle L,\phi_\alpha\rangle\!\rangle_{0,2} = \sum_{d,l,n}\frac{Q^d}{l!}\langle L,\phi_\alpha,\tau,\dots,\tau;t,\dots,t\rangle^{S_n}_{0,2+l+n,d} = \\ \langle L,\phi_\alpha,\tau+t\rangle_{0,3,0}+\sum_{d,l,n}\frac{Q^d}{l!}\langle\phi_\alpha,\tau+t,\tau,\dots,\tau;t,\dots,t\rangle^{S_n}_{0,2+l+n} = \\ (\tau+t,\phi_\alpha)+\langle\!\langle\tau+t,\phi_\alpha\rangle\!\rangle_{0,2},$$

and hence $\sum_{\alpha,\beta}\langle\!\langle L,\phi_\alpha\rangle\!\rangle_{0,2}G^{\alpha\beta}\phi_\beta = t+\tau$.

Finally, using the dilaton equations

$$\langle\!\langle L-1,A\rangle\!\rangle_{0,2} = -\langle\!\langle A\rangle\!\rangle_{0,1}+\langle\!\langle A,t+\tau\rangle\!\rangle_{0,2}, \\ \langle\!\langle L-1\rangle\!\rangle_{0,1} = -2\langle\!\langle\ \rangle\!\rangle_{0,0}+\langle\!\langle t+\tau\rangle\!\rangle_{0,1},$$

we find that

$$\langle\!\langle\ \rangle\!\rangle_{0,0}+\langle\!\langle\mathbf{y}+L-1-t-\tau\rangle\!\rangle_{0,1}+\frac{1}{2}\langle\!\langle\mathbf{y}+L-1-t-\tau,\mathbf{y}+L-1-t-\tau\rangle\!\rangle_{0,2}$$

transforms into $\langle\!\langle\mathbf{y},\mathbf{y}\rangle\!\rangle_{0,2}/2$. Indeed, the terms linear in $\mathbf{y}$

$$\langle\!\langle\mathbf{y}\rangle\!\rangle_{0,1}+\langle\!\langle L-1-t-\tau,\mathbf{y}\rangle\!\rangle_{0,2} = \langle\!\langle\mathbf{y}\rangle\!\rangle_{0,1}-\langle\!\langle\mathbf{y}\rangle\!\rangle_{0,1}+\langle\!\langle\mathbf{y},t+\tau\rangle\!\rangle_{0,2}-\langle\!\langle t+\tau,\mathbf{y}\rangle\!\rangle_{0,2}$$

cancel out. The $\mathbf{y}$-independent terms

$$\frac{1}{2}\langle\langle L-1,L-1\rangle\rangle_{0,2}-\langle\langle L-1,t+\tau\rangle\rangle_{0,2}+\frac{1}{2}\langle\langle t+\tau,t+\tau\rangle\rangle_{0,2}$$
$$+\langle\langle L-1\rangle\rangle_{0,1}-\langle\langle t+\tau\rangle\rangle_{0,1}+\langle\langle\ \rangle\rangle_{0,0}=-\frac{1}{2}\langle\langle L-1,t+\tau\rangle\rangle_{0,2}+$$
$$\frac{1}{2}\langle\langle t+\tau,t+\tau\rangle\rangle_{0,2}+\frac{1}{2}\langle\langle L-1\rangle\rangle_{0,1}-\langle\langle t+\tau\rangle\rangle_{0,1}+\langle\langle\ \rangle\rangle_{0,0}=$$
$$\frac{1}{2}\langle\langle t+\tau\rangle\rangle_{0,1}+\frac{1}{2}\langle\langle t+\tau\rangle\rangle_{0,1}-\langle\langle\ \rangle\rangle_{0,0}-\langle\langle t+\tau\rangle\rangle_{0,1}+\langle\langle\ \rangle\rangle_{0,0}.$$

cancel out too. Thus, we obtain

$$\mathcal{F}_g(\mathbf{y}+q-1-t)=\bar{\mathcal{F}}_g([S\mathbf{y}]_+ +q-1)+\delta_{g,1}\langle\langle\ \rangle\rangle_{1,0}+\frac{\delta_{g,0}}{2}\langle\langle \mathbf{y}(L),\mathbf{y}(L)\rangle\rangle_{0,2}.$$

In view of Proposition 5 from the previous section, we have proved the following theorem.

`Theorem 1.` *The total descendent potential after the shift by $1-q+t$:*

$$\mathcal{D}(1-q+t+\mathbf{x})=e^{\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_g(\mathbf{x})},$$

and the τ-family of total ancestor potentials after the shift by $1-q$:

$$\mathcal{A}_\tau(1-q+\mathbf{x}):=e^{\sum_{g\geq 0}\hbar^{g-1}\bar{\mathcal{F}}_g^{(\tau)}(\mathbf{x})},\ \tau\in K^0(X)\otimes\Lambda,$$

are related by the family of quantized operators

$$\mathcal{D}=e^{F_1(\tau)}\,\widehat{S_\tau^{-1}}\,\mathcal{A}_\tau,$$

where $F_1(\tau)=\langle\langle\ \rangle\rangle_{1,0}:=\sum_{k,n,d}\frac{Q^d}{k!}\langle\tau,\dots,\tau;t,\dots,t\rangle_{1,k+n,d}^{S_n}$ is the generating function for primary GW-invariants of genus 1.

Passing to the quasi-classical limit $\hbar\to 0$, one obtains

`Corollary 1.` *The graph $\mathcal{L}\subset\mathcal{K}$ of the differential of the genus-0 descendent potential $\mathcal{F}_0(\mathbf{y}+q-1-t)$ and the τ-family $\mathcal{L}^{(\tau)}\subset\mathcal{K}$ of the graphs of the differentials of genus-0 ancestor potentials $\bar{F}_0^{(\tau)}(\mathbf{y}+q-1)$ are related by symplectic transformations $S_\tau:(\mathcal{K},\Omega)\to(\mathcal{K},\bar{\Omega}^{(\tau)})$:*

$$\mathcal{L}=S_\tau^{-1}\mathcal{L}^{(\tau)}.$$

The genus-0 ancestor correlators $\langle\langle\mathbf{x}(\bar{L}),\dots,\mathbf{x}(\bar{L})\rangle\rangle_{0,m}$ have the "zero 2-get" property [5]; namely they have zero 2-jet along the subspace $\mathbf{x}\in\mathcal{K}_+$, where $\mathbf{x}(1)=0$. This is because $\bar{L}_i$ are pull-backs of the line bundles L_i from the Deligne-Mumford space $\overline{\mathcal{M}}_{0,m}$, which is a *manifold* of dimension $m-3$, and where therefore any product of $m-2$ factors $\bar{L}_i-1$ vanishes for dimensional reasons. Since the dilaton shift $1-q$ also vanishes at $q=1$, we conclude that $\mathcal{L}^{|tau}$ *is tangent to* $\mathcal{K}_+$ *along* $(1-q)\mathcal{K}_+$. *Consequently,* $T_{|tau}:=S_\tau^{-1}\mathcal{K}_+$ *is tangent to* $\mathcal{L}$ *along*

$(1-q)T_\tau \subset \mathcal{L}$. In fact, as τ varies, these spaces sweep $\mathcal{L}$ (which is easy to check modulo Novikov's variables, and then apply the formal Implicit Function Theorem.)

`Corollary 2.` *$\mathcal{L} \subset (\mathcal{K}, \Omega)$ is an overruled Lagrangian cone, i.e. its tangent spaces $T := T_{\mathcal{J}}\mathcal{L}$ are tangent to $\mathcal{L}$ exactly along $(1-q)T \subset \mathcal{L}$.*

Example: $X = pt$

In Part I, we found that for $t \in \Lambda$

$$\mathcal{J}(0,t) := 1-q+t+\sum_{n\geq 2}\langle \frac{1}{1-qL}; t,\dots,t\rangle_{0,1+n}^{S_n} = (1-q)e^{\sum_{k>0}\Psi^k(t)/k(1-q^k)}.$$

It follows from the string equation (see Corollary 2 of Proposition 4) that for $\tau \in \Lambda$

$$\mathcal{J}(\tau,t) = 1-q+t+\tau+\langle\langle \frac{1}{1-qL}\rangle\rangle_{0,1} = (1-q)e^{\tau/(1-q)+\sum_{k>0}\Psi^k(t)/k(1-q^k)}.$$

Taking $q=0$ (and using the string equation twice), we find the variable metric

$$G(\tau) = G_{11}(\tau) := \langle\langle 1,1,1\rangle\rangle_{0,3} = 1+\tau+t+\langle\langle 1\rangle\rangle_{0,1} = e^{\tau+\sum_{k>0}\Psi^k(t)/k}.$$

Using the string equation once more, we derive that

$$\begin{aligned} S_\tau(q) &:= \left(1+\langle\langle \frac{1}{1-L/q}, 1\rangle\rangle_{0,2}\right) G^{-1}(\tau) = \frac{\mathcal{J}(1/q)}{1-1/q}\, G^{-1}(\tau) \\ &= e^{\tau/(q-1) - \sum_{k>0}\Psi^k(t)/k(q^k-1)}, \\ S_\tau^{-1}(q) &= e^{\tau/(1-q) + \sum_{k>0}\Psi^k(t)/k(1-q^k)} = \frac{\mathcal{J}(q)}{1-q}, \end{aligned}$$

and find the range $\mathcal{L}_t$ of the J-function $\mathcal{K}_+ \to \mathcal{K} : \mathbf{x} \mapsto \mathcal{J}(\mathbf{x},t)$ to be

$$\mathcal{L}_t = \bigcup_{\tau\in\Lambda} e^{\tau/(1-q) + \sum_{k>0}\Psi^k(t)/k(1-q^k)}(1-q)\mathcal{K}_+.$$

At $\tau = 0$, we have here one of the subspace in $\mathcal{K}$, depending on t, whose union over $t \in \Lambda_+$, according to the results of Part III, yields the range $\mathcal{L}$ of the permutation-equivariant J-function $\mathbf{t} \mapsto \mathcal{J}(0,\mathbf{t})$

$$\mathcal{L} = \bigcup_{t\in\Lambda_+} e^{\sum_{k>0}\Psi^k(t)/k(1-q^k)}(1-q)\mathcal{K}_+.$$

In fact this picture remains true in general, as we will now show.

Adelic characterization

We return now to the mixed genus-0 descendent potential $\mathcal{F}_0(\mathbf{x},\mathbf{t})$ with the permutable input $\mathbf{t}\in\mathcal{K}_+$ allowed to involve the cotangent line bundles L_i. In the symplectic loop space $(\mathcal{K},\Omega)$, it is represented by the dilaton-shifted graph of its differential. According to Proposition 4 and its Corollary 1, it is the range of the J-function

$$\mathcal{K}_+\ni\mathbf{x}\mapsto\mathcal{J}(\mathbf{x},\mathbf{t}):=1-q+\mathbf{t}(q)+\mathbf{x}(q)+ \\ \sum_{\alpha,k,n,d}\phi^\alpha\frac{Q^d}{k!}\langle\frac{\phi_\alpha}{1-qL},\mathbf{x}(L),\dots,\mathbf{x}(L);\mathbf{t}(L),\dots,\mathbf{t}(L)\rangle_{0,1+k+n,d}^{S_n},$$

and has the form of a Lagrangian cone $\mathcal{L}_{\mathbf{t}}$, depending on the parameter $\mathbf{t}\in\mathcal{K}_+$. According to the results of the previous section $\mathcal{L}_{\mathbf{t}}$ is an overruled Lagrangian cone whenever $\mathbf{t}$ is constant in q. We combine this information with the *adelic characterization* of the J-function given in [8, 11, 12][1] and discussed in Part III, to prove the following theorem.

`Theorem 2.` *The range $\mathcal{L}$ of permutation-equivariant J-function $\mathbf{t}\mapsto\mathcal{J}(0,\mathbf{t})$ (with $\mathbf{t}\in\mathcal{K}_+$, and $\mathbf{t}(1)\in K^0(X)\otimes\Lambda_+$, where Λ_+ is a certain neighborhood of $0\in\Lambda$) has the form*

$$\mathcal{L}=\bigcup_{t\in K^0(X)\otimes\Lambda_+}(1-q)\,S_0^{-1}(q)_t\,\mathcal{K}_+,$$

where the operators $S_\tau(q)$ evaluated at $\tau=0$ still depend on the parameter $t\in K^0(X)\otimes\Lambda_+$:

$$S_0^{-1}(q)_t\,\psi:=\psi+\sum_\alpha\phi_\alpha\sum_{n,d}Q^d\langle\psi,\frac{\phi_\alpha}{1-qL};t,\dots,t\rangle_{0,2+n,d}^{S_n}.$$

`Proof.` According to the adelic characterization results, *a rational function $\mathbf{f}\in\mathcal{K}$ lies in $\mathcal{L}_{\mathbf{t}}$ if and only if its Laurent series expansions $\mathbf{f}_{(\zeta)}$ near $q=1/\zeta$ satisfy the following three conditions:*

(i) $\mathbf{f}_{(1)}\in\mathcal{L}^{fake}\subset\widehat{\mathcal{K}}$, the range, in the space $\widehat{\mathcal{K}}$ of Laurent series in $q-1$ with vector coefficients in $\mathcal{K}^0(X)\otimes\Lambda$, of the J-function in the fake *quantum K-theory of X;*

(ii) when $\zeta\neq 0,1,\infty$ is a primitive mth root of unity, $\mathbf{f}_{(\zeta)}(q^{1/m}/\zeta)\in\mathcal{L}_{\mathbf{t}}^{(\zeta)}$, a certain Lagrangian subspace in $\widehat{K}$ which will be specified below;

(iii) when $\zeta\neq 0,\infty$ is not a root of unity, $\mathbf{f}_{(\zeta)}$ is a power series in $q-1/\zeta$, i.e. $\mathbf{f}$ has no pole at $q=1/\zeta$.

---

[1] Formally speaking, there only the case $\mathbf{t}=0$ is considered, but the results extend without change to the general case, where the moduli orbi-spaces are $X_{0,1+k+n,d}/S_n$ rather than $X_{0,1+k,d}$.

To elucidate the situation, recall that in *fake* K-theory, the genuine holomorphic Euler characteristics $\chi(\mathcal{M}; V)$ are replaced with their "fake" values given by the right-hand-side of the Hirzebruch–Riemann–Roch formula:

$$\chi^{fake}(\mathcal{M}; V) := \int_{[\mathcal{M}]} \operatorname{ch}(V)\, \operatorname{td}(T_{\mathcal{M}}).$$

Fake in this sense GW-invariants were studied, e.g. in [2]. In particular, the range of the fake J-function is known to be an overruled Lagrangian cone $\mathcal{L}^{\text{fake}} \subset (\widehat{\mathcal{K}}, \widehat{\Omega})$, where $\widehat{\Omega}(\mathbf{f}, \mathbf{g}) = \operatorname{Res}_{q=1}(\mathbf{f}(q^{-1}), \mathbf{g}(q))\, q^{-1} dq$.

The moduli spaces of stable maps behave as virtual *orbifolds* (rather than manifolds), and the genuine holomorphic Euler characteristics are given by the virtual Kawasaki–Riemann–Roch formula [11], summing up certain fake holomorphic Euler characteristics of the inertia orbifold (of the moduli spaces $X_{0,1+k+n}/S_n$ in our situation). Figure 3, essentially copied from Part III, is to remind us of the recursive device keeping track of all Kawasaki contributions into the J-function.

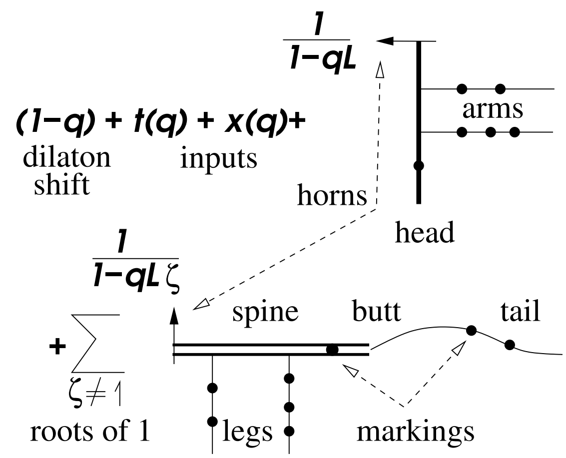

FIGURE 3. Adelic characterization

In particular, it shows that the values of the J-function, when expanded into near $q = 1$, lie in $\mathcal{L}^{fake}$, and when expanded near a primitive mth root of 1, they are characterized in terms of certain twisted fake invariants of the orbifold target space $X \times B\mathbb{Z}_m$. The latter, in their turn, are expressed in terms of the untwisted fake invariants of X. Namely in the test (ii) above, the subspace $\mathcal{L}_{\mathbf{t}}^{(\zeta)} \subset \widehat{K}$ is obtained from *a certain tangent space $T_{\mathbf{t}}^{fake}$ to $\mathcal{L}^{fake}$ by the linear transformation:*

$$\mathcal{L}_{\mathbf{t}}^{(\zeta)} = e^{\sum_{k>0} \left(\frac{\Psi^k(T_X^*)}{k(1-\zeta^{-k} q^{k/m})} - \frac{\Psi^{km}(T_X^*)}{k(1-q^{km})} \right)} \Psi^m(T_{\mathbf{t}}^{fake}) \otimes_{\Psi^m(\Lambda)} \Lambda.$$

In our present discussion, it is important to figure out what determines the application point of the tangent space $T^{fake}_{\mathbf{t}}$. On the diagram, it is determined by *legs*, which are related by the Adams operation Ψ^m to *arms* (see Part III, or [8]). Note however, that the markings on the legs (each representing m copies of markings on the arms attached to the m-fold cover of the spine curve) are allowed to carry permutable inputs $\mathbf{t}$, but not allowed to carry the non-permutable inputs $\mathbf{x}$, because their numbering would break the $\mathbb{Z}_m$-symmetry of the covering curve). Consequently, *the value of $\mathcal{J}^{fake} \in \mathcal{L}^{fake}$, which determines the application point of the tangent space $T_{\mathbf{t}}$, is obtained by the expansion near $q = 1$ of the J-function with the non-permutable input $\mathbf{x} = 0$: $T_{\mathbf{t}} = T_{\mathcal{J}(0,\mathbf{t})_{(1)}}\mathcal{L}^{fake}$.* In fact, since $\mathcal{L}^{fake}$ is overruled, its tangent spaces to $\mathcal{L}^{fake}$ are parameterized by $K^0(X) \otimes \Lambda$. Let us analyze the map $\mathbf{t} \mapsto$ (tangent space to $\mathcal{L}^{fake}$).

In degree $d = 0$, the J-function of X coincides with the J-function of the point target space with coefficients in the λ-algebra $\Lambda' := K^0(X) \otimes \Lambda$. It was described in section Example. For $\mathbf{t} = t \in \Lambda'$ (i.e. q-independent), we have

$$\mathcal{J}(0,t)_{(1)} = (1-q)e^{\sum_{k>0}\Psi^k(t)/k^2(1-q)} \times (\text{power series in } q-1).$$

In other words, $\sum_{k>0}\Psi^k(t)/k^2$ is the parameter value of the tangent space to $\mathcal{L}^{fake}$ associated to the input $t \in \Lambda'$ in this approximation.

The series is not guaranteed to converge. E.g. under the identification of $K^0(X) \otimes \mathbb{Q}$ with $H^{even}(X, \mathbb{Q})$ by the Chern character, Ψ^k acts on $H^{2r}(X)$ as multiplication by k^r, and the series $\sum_{k>0} k^{r-2}$ diverges unless $r = 0$. To handle this difficulty, we assume that the ground ring Λ is topologized with a filtration $\Lambda \supset \Lambda_+ \supset \Lambda_{++} \supset \dots$ by ideals such that Ψ^k with $k > 1$ increase the filtration. For instance, when $\Lambda = \mathbb{Q}[[Q]]$ is the Novikov ring, $\Psi^k(Q^d) = Q^{kd}$, the filtration by the powers of the maximal ideal is taken. When $\Lambda = \mathbb{Q}[[N_1, N_2, \dots]]$ is the ring of symmetric functions, $\Psi^k(N_r) = N_{kr}$, the filtration by degrees of symmetric functions suffices. Then the map $t \mapsto \sum_{k>0}\Psi^k(t)/k^2$ converges for $t \in K^0(X) \otimes \Lambda_+$, and is invertible in this range,[2] since $\Psi^1(t) = t$.

Returning to the general input $\mathbf{t}$ and degree $d \geq$, we conclude from the formal Implicit Function Theorem, that there is a well-defined map

$$\mathcal{T} : \{\mathbf{t} \in \mathcal{K}_+ \,|\, \mathbf{t}(1) \in K^0(X) \otimes \Lambda_+\} \to K^0(X) \otimes \Lambda_+,$$

such that $T_{\mathcal{J}(0,\mathbf{t})_{(1)}}\mathcal{L}^{fake} = T_{\mathcal{J}(0,\mathcal{T}(\mathbf{t}))_{(1)}}\mathcal{L}^{fake}$. For all inputs $\mathbf{t}$ with the same value $\mathcal{T}(\mathbf{t})$, the adelic characterization tests (i), (ii), (iii) coincide.

[2] Even in the entire $H^0(X, \Lambda)$, if $\sum_{k>0} k^{-2} = \pi^2/6$ is adjoined to Λ.

By the same token, for each t there is a well-defined map $\mathcal{K}_+ \to K^0(X) \otimes \Lambda : \mathbf{x} \mapsto \tau(\mathbf{x})$, such that $T_{\mathcal{J}(\mathbf{x},t)}\mathcal{L}_t = T_{\mathcal{J}(\tau(\mathbf{x}),t)}\mathcal{L}_t$. For all inputs $\mathbf{x}$ with the same $\tau(\mathbf{x})$, the values $\mathcal{J}(\mathbf{x},t)$ of the J-function form the ruling space $(1-q)\, S_\tau(q)_t\, \mathcal{K}_+$ of the overruled cone $\mathcal{L}_t$. For all such points, the localizations $\mathcal{J}(\mathbf{x},\mathbf{t})_{(1)}$ lie in the same ruling space of $\mathcal{L}^{fake}$, and moreover, when $\tau = 0$, the last ruling space is the one where $\mathcal{J}(0,\mathbf{t})$ with $\mathcal{T}(\mathbf{t}) = t$ lie. Thus, for rational functions from the space $(1-q)\, S_0(q)_t\, \mathcal{K}_+$ and for the values $\mathcal{J}(0,\mathbf{t})$ with $\mathcal{T}(\mathbf{t}) = t$, the adelic characterization tests (i), (ii), (iii) coincide, i.e. the localizations in test (i) lie in the same ruling space of $\mathcal{L}^{fake}$, and the tangent spaces to $\mathcal{L}^{fake}$ involved into test (ii) are the same. Therefore the two sets of rational functions coincide:

$$\{\mathcal{J}(0,\mathbf{t}) \,|\, \mathcal{T}(\mathbf{t}) = t\} = (1-q)\, S_0(q)_t\, \mathcal{K}_+.$$

Taking the union over $t \in \Lambda_+$ completes the proof. □

`Corollary 1.` *$\mathcal{L}_\mathbf{t} = \mathcal{L}_t$, where $t = \mathcal{T}(\mathbf{t})$.*

`Corollary 2.` *Each $\mathcal{L}_\mathbf{t}$ is an overruled Lagrangian cone invariant under the string flow $\mathbf{f} \mapsto e^{\epsilon/(1-q)}\mathbf{f}$, $\epsilon \in \Lambda$.*

`Remark.` The range $\mathcal{L} \subset (\mathcal{K}, \Omega)$ of the permutation-equivariant J-function $\mathbf{t} \mapsto \mathcal{J}(0,\mathbf{t})$ is a cone ruled by the family $t \mapsto R_t := (1-q)S_0(q)_t\mathcal{K}_+$ of *isotropic* subspaces (and is in this sense "overruled") but it is not Lagrangian, nor is it invariant under the string flow, as the example of $X = pt$ readily illustrates. In particular, the spaces $R_t/(1-q)$ are not tangent to $\mathcal{L}$, and do not form semi-infinite variations of Hodge structures in the sense of S. Barannikov [1]. Nevertheless from Proposition 2 (dilaton equation), we have:

`Corollary 3.` *The permutation-equivariant genus-0 descendent potential*

$$\mathcal{F}_0(0,\mathbf{t}) := \sum_{0,n,d} Q^d \langle \mathbf{t}(L), \dots, \mathbf{t}(L)\rangle_{0,n,d}^{S_n}$$

is reconstructed from the permutation-equivariant J-function by

$$\frac{1}{2}\Omega\left([\mathcal{J}(0,\mathbf{t})]_-, [\mathcal{J}(0,\mathbf{t})]_+\right) = \mathcal{F}_0(0,\mathbf{t}) + \frac{(\Psi^2(\mathbf{t}(1)), 1)}{2}.$$

References

[1] S. Barannikov. *Quantum periods. I. Semi-infinite variations of Hodge structures.* Internat. Math. Res. Notices 23 (2001), 1243–1264.

[2] T. Coates. *Riemann–Roch theorems in Gromov–Witten theory.* PhD thesis, 2003, available at http://math.harvard.edu/ tomc/thesis.pdf

[3] T. Coates, A. Givental. *Quantum Riemann–Roch, Lefschetz and Serre.* Ann. of Math. (2), 165 (2007), 15-53.

[4] E. Getzler. *Topological recursion relations in genus 2.* "Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)." World Sci. Publishing, River Edge, NJ, 1998, pp 73–106.

[5] E. Getzler. *The jet-space of a Frobenius manifold and higher-genus Gromov–Witten invariants.* Frobenius Manifolds, Aspects of Mathematics, Volume 36, 2004, pp 45-89.

[6] A. Givental. *On the WDVV-equation in quantum K-theory.* Mich. Math. J. 48 (2000) 295-304.

[7] A. Givental. *Gromov-Witten invariants and quantization of quadratic Hamiltonians.* Mosc. Math. J. 1 (2001), no. 4, 551–568, 645 (English, with English and Russian summaries).

[8] A. Givental, V. Tonita. *The Hirzebruch-Riemann–Roch theorem in true genus-0 quantum K-theory.* Preprint, arXiv:1106.3136

[9] T. Kawasaki. *The Riemann-Roch theorem for complex V-manifolds.* Osaka J. Math. Volume 16, Number 1 (1979), 151-159.

[10] M. Kontsevich, Yu. Manin. *Relations between the correlators of the topological sigma-model coupled to gravity.* Comm. Math. Phys. 196 (1998), no. 2, 385–398.

[11] V. Tonita. *A virtual Kawasaki Riemann–Roch formula.* Pacific J. Math. 268 (2014), no. 1, 249–255. arXiv:1110.3916.

[12] V. Tonita. *Twisted orbifold Gromov–Witten invariants.* Nagoya Math. J. 213 (2014), 141–187, arXiv:1202.4778