PERMUTATION-EQUIVARIANT QUANTUM K-THEORY IV. $\mathcal{D}_q$-MODULES

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Abstract. In Part II, we saw how permutation-equivariant quantum K-theory of a manifold with isolated fixed points of a torus action can be reduced via fixed point localization to permutation-equivariant quantum K-theory of the point. In Part III, we gave a complete description of permutation-equivariant quantum K-theory of the point by means of adelic characterization. Here we apply the adelic characterization to introduce the action on this theory of a certain group of q-difference operators. This action enables us to prove that toric q-hypergeometric functions represent K-theoretic GW-invariants of toric manifolds.

Overruled cones and $\mathcal{D}_q$-modules

In Part III, we gave the following *adelic characterization* of the big J-function $\mathcal{J}_{pt}$ of the point target space. In the space $\mathcal{K}$ of "rational functions" of q (consisting in fact of series in auxiliary variables with coefficients which are rational functions of q), let $\mathcal{L}$ denote the range of $\mathcal{J}_{pt}$. We showed that *an element $f \in \mathcal{K}$ lies in $\mathcal{L}$ if and only if Laurent series expansions $f_{(\zeta)}$ of f near $q = \zeta^{-1}$ satisfy*

(i) $f_{(1)} = (1-q)e^{\tau/(1-q)} \times$ (power series in $q-1$) *for some* $\tau \in \Lambda_+$,[1]

(ii) when $\zeta \neq 1$ is a primitive m-th root of unity,

$$f_{(\zeta)}(q^{1/m}/\zeta) = \Psi^m(f_{(1)}/(1-q)) \times (\text{power series in } q-1),$$

where Ψ^m is the Adams operation extended from Λ by $\Psi^m(q) = q^m$;

(iii) when $\zeta \neq 0, \infty$ is not a root of unity, $f_{(\zeta)}(q/\zeta)$ is a power series in $q-1$.

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[1] For convergence purposes, we assume that the Adams operations Ψ^k on Λ with $k > 1$ increase certain filtration $\Lambda \supset \Lambda_+ \supset \Lambda_{++} \supset \cdots$, and that the domain of the J-function is Λ_+.

Another way to phrase (i) is to say that $f_{(1)}$ lies in the range $\mathcal{L}^{fake}$ of the *ordinary* (or *fake*) J-function $\mathcal{J}_{pt}^{ord}$ in the space $\widehat{\mathcal{K}} = \Lambda((q-1))$ of Laurent series in $q-1$:

$$\mathcal{L}^{fake} = \bigcup_{\tau \in \Lambda_+} (1-q) e^{\tau/(1-q)} \widehat{K}_+, \quad \widehat{K}_+ := \Lambda[[q-1]].$$

The range $\mathcal{L}^{\text{fake}}$ is an example of an *overruled cone*: Its tangent spaces $T_\tau = e^{\tau/(1-q)}\widehat{\mathcal{K}}_+$ are tangent to $\mathcal{L}^{fake}$ along the subspaces $(1-q)T_\tau$ (which sweep $\mathcal{L}^{fake}$ as the parameter τ varies through Λ_+.[2]) As it will be explained shortly, this property leads to the invariance of $\mathcal{L}^{fake}$ to certain finite-difference operators.

Recall that in permutation-equivariant quantum K-theory, we work over a λ-algebra, a ring equipped with Adams homomorphisms Ψ^m, $m = 1, 2, \dots,$ $\Psi^1 = \mathrm{Id}$, $\Psi^m \Psi^l = \Psi^{ml}$. Let us take $\Lambda := \Lambda_0[[\lambda, Q]]$ with $\Psi^m(\lambda) = \lambda^m$, $\Psi^m(Q) = Q^m$, where Λ_0 is any ground λ-algebra over $\mathbb{C}$.

Consider the algebra of finite-difference operators in Q. Such an operator is a non-commutative expression $D(Q, 1-q^{Q\partial_Q}, q^{\pm 1})$. Clearly, the space $\widehat{\mathcal{K}}_+ = \Lambda[[q-1]]$ (as well as $(1-q)\widehat{\mathcal{K}}_+$) is a $\mathcal{D}_q$-module. Consequently each ruling space $(1-q)T_\tau = e^{\tau/(1-q)}(1-q)\widehat{K}_+$ is a $\mathcal{D}_q$-module too. Indeed,

$$q^{Q\partial_Q} e^{\tau(Q)/(1-q)} = e^{\tau(Q)/(1-q)} e^{(\tau(qQ)-\tau(Q))/(1-q)},$$

where the second factor lies in $\widehat{\mathcal{K}}_+$. Moreover, we have

`Proposition.` $e^{\lambda D(Q, 1-q^{Q\partial_Q}, q)/(1-q)} \mathcal{L}^{fake} = \mathcal{L}^{fake}$.

`Proof.` The ruling space $(1-q)T_\tau$ is a $\mathcal{D}_q$-module, and hence invariant under D. Therefore for $f \in (1-q)T_\tau$, we have $Df/(1-q) \in T_\tau$, i.e. the vector field defining the flow $t \mapsto e^{t\lambda D/(1-q)}$ is tangent to $\mathcal{L}^{fake}$, and so the flow preserves $\mathcal{L}^{fake}$. It remains to take $t = 1$, which is possible thanks to λ-adic convergence.

`Remark.` Generally speaking, linear transformation $e^{\lambda D/(1-q)}$ does not preserve ruling spaces $(1-q)T_\tau$, but transforms each of them into another such space. Indeed, preserving $\mathcal{L}^{fake}$, it transform tangent spaces T_τ into tangent spaces, and since it commutes with multiplication by $1-q$, it also transforms ruling spaces $(1-q)T_\tau$ into ruling spaces.

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[2]In terminology of S. Barannikov [1], this is a variation of semi-infinite Hodge structures: The flags $\cdots \subset (1-q)T_\tau \subset T_\tau \subset (1-q)^{-1}T_\tau \subset \cdots$ vary in compliance with "Griffiths' transversality condition".

Likewise, cone $\mathcal{L} \subset \mathcal{K}$ is ruled by subspaces comparable to $(1-q)\mathcal{K}_+$, namely by $(1-q)L_\tau$, where $L_\tau := e^{\sum_{k>0} \Psi^k(\tau)/k(1-q^k)} \mathcal{K}_+$. However L_τ are not tangent to $\mathcal{L}$. Nonetheless the following result holds.

`Theorem.` *The range $\mathcal{L}$ of the big J-function $\mathcal{J}_{pt}$ in the permutation-equivariant quantum K-theory of the point target space is preserved by operators of the form*

$$e^{\sum_{k>0} \lambda^k \Psi^k\left(D(1-q^{kQ\partial_Q}, q^{\pm 1})\right)/k(1-q^k)}.$$

`Remarks.` (1) The operator D has constant coefficients, i.e. is independent of Q.

(2) Note that $\Psi^k(q^{Q\partial_Q}) = q^{kQ^k\partial_{Q^k}} = q^{Q\partial_Q}$, and not $q^{kQ\partial_Q}$ as in the exponent.

(3) The reader is invited to check that the theoem and its proof are extended without any changes to the case finite difference operators in several variables $Q_1, \dots, Q_K$. We will use the theorem in this more general form in Part V.

`Proof.` Assuming that $(1-q)f \in \mathcal{L}$, we use the adelic characterization of $\mathcal{L}$ to show that $(1-q)g \in \mathcal{L}$, where

$$g(q) := e^{\sum_{k>0} \lambda^k \Psi^k\left(D(1-q^{kQ\partial_Q}, q^{\pm 1})\right)/k(1-q^k)} f(q).$$

First, this relationship between g and f also holds between $g_{(1)}$ and $f_{(1)}$ where however both sides need to be understood as Laurent series in $q-1$. Since $f_{(1)} \in \mathcal{L}^{fake}$, Proposition implies that $g_{(1)} \in \mathcal{L}^{fake}$ too.

Next, applying Ψ^m to both sides, we find

$$\Psi^m(g_{(1)}) = e^{\sum_{l>0} \lambda^{ml} \Psi^{ml}\left(D(1-q^{lQ\partial_Q}, q^{\pm 1})\right)/l(1-q^{ml})} \Psi^m(f_{(1)}).$$

On the other hand, for an m-th primitive root of unity ζ, taking into account that $\Psi^{ml}(q) = q^{ml}$ turns after the change $q \mapsto q^{1/m}/\zeta$ into q^l, and that $q^{mlQ\partial_Q}$ turns after this change into $q^{lQ\partial_Q}$, we find

$$g_{(\zeta)}(q^{1/m}/\zeta) = e^{\triangle} e^{\sum_{l>0} \lambda^{ml} \Psi^{ml}\left(D(1-q^{lQ\partial_Q}, q^{\pm 1/m})\right)/ml(1-q^l)} f_{(\zeta)}(q^{1/m}/\zeta),$$

where the finite-difference operator $\triangle$ has coefficients regular at $q = 1$. Here we factor off the terms regular at $q=1$ using the fact that our operators have constant coefficients, and hence commute. Namely, $e^{A+B/(1-q)}$, where A and B are regular at $q=1$, can be rewritten as $e^A e^{B/(1-q)}$.

We are given that $f_{(\zeta)}(q^{1/m}/\zeta) = p\, \Psi^m(f_{(1)})$ where $p \in \widehat{\mathcal{K}}_+$. Since $[q^{Q\partial_Q}, Q] = (q-1)Qq^{Q\partial_Q}$ is divisible by $q-1$, for any finite-difference operator B, the commutator $\mathrm{ad}_B(p) = [B, p]$ with the operator of multiplication by p is divisible by $q-1$. Therefore $e^{B/(1-q)}p = Pe^{B/(1-q)}$, where $P = e^{\mathrm{ad}_{B/(1-q)}}(p)$ is regular at $q = 1$. Thus, for some P regular at $q = 1$ we have:

$$g_{(\zeta)}(q^{1/m}/\zeta) = e^{\triangle} P e^{\sum_{l>0} \lambda^{ml} \Psi^{ml}\left(D(1-q^{lQ\partial_Q}, q^{\pm 1/m})\right)/ml(1-q^l)}\, \Psi^m(f_{(1)}).$$

Comparing this expression with $\Psi^m(g_{(1)})$, take into account that $q^{\pm 1/m}$ coincides with $q^{\pm 1}$ modulo $q-1$, and $1/(1-q^{-lm}) - 1/m(1-q^{-l})$ is regular at $q = 1$. Thus, again factoring off the terms regular at $q = 1$, we conclude that $g_{(\zeta)}(q^{1/m}/\zeta)$ is obtained from $\Psi^m(g_{(1)})$ by the application of an operator regular at $q = 1$.

From the explicit description of $\mathcal{L}^{fake}$, we have $g_{(1)} \in e^{\tau/(1-q)}\widehat{\mathcal{K}}_+$ for some τ. Therefore $\Psi^m(g_{(1)}) \in e^{\Psi^m(\tau)/m(1-q)}\widehat{\mathcal{K}}_+$. The latter is a $\mathcal{D}_q$-module, and hence $g_{(\zeta)}(q^{1/m}/\zeta) \in \Psi^m(g_{(1)})\,\widehat{\mathcal{K}}_+$ as required.

Finally, for $\zeta \neq 0, \infty$, which is not a root of unity, regularity of g at $q = \zeta^{-1}$ is obvious whenever the same is true for f. $\square$

Γ-operators

`Lemma.` *Let l be a positive integer. Suppose that $\sum_{d\geq 0} f_d Q^d$ represents a point on the cone $\mathcal{L} \subset \mathcal{K}$. Then the same is true about:*

$$\sum_{d\geq 0} f_d\, Q^d \prod_{r=0}^{ld-1}(1-\lambda q^{-r}),\ \sum_{d\geq 0} \frac{f_d\, Q^d}{\prod_{r=1}^{ld}(1-\lambda q^r)},\ \textit{and} \sum_{d\geq 0} f_d\, Q^d \prod_{r=1}^{ld}(1-\lambda q^r).$$

`Proof.` We use q-Gamma-function

$$\Gamma_q(x) := e^{\sum_{k>0} x^k/k(1-q^k)} \sim \prod_{r=0}^{\infty} \frac{1}{1-xq^r}$$

for symbols of q-difference operators:

$$\frac{\Gamma_{q^{-1}}(\lambda q^{-lQ\partial_Q})}{\Gamma_{q^{-1}}(\lambda)}\, Q^d = Q^d\, \frac{\prod_{r=-\infty}^{0}(1-\lambda q^r)}{\prod_{r=-\infty}^{-ld}(1-\lambda q^r)} = Q^d \prod_{r=0}^{ld-1}(1-\lambda q^{-r}),$$

$$\frac{\Gamma_{q^{-1}}(\lambda q^{lQ\partial_Q})}{\Gamma_{q^{-1}}(\lambda)}\, Q^d = Q^d\, \frac{\prod_{r=-\infty}^{0}(1-\lambda q^r)}{\prod_{r=-\infty}^{ld}(1-\lambda q^r)} = \frac{Q^d}{\prod_{r=1}^{ld}(1-\lambda q^r)}, \text{ and}$$

$$\frac{\Gamma_{q^{-1}}(\lambda)}{\Gamma_{q^{-1}}(\lambda q^{lQ\partial_Q})}\, Q^d = Q^d \prod_{r=1}^{ld}(1-\lambda q^r) \text{ respectively.}$$

The result follows now from the theorem of the previous section. $\square$

Application to fixed point localization

In Part II, we used fixed point localization to characterize the range (denote it $\mathcal{L}_X$) of the big J-function in permutation- (and torus-) equivariant quantum K-theory of $X = \mathbb{C}P^N$. Namely *a vector-valued "rational function" $f(q) = \sum_{i=0}^N f^{(i)}(q)\phi_i$ represents a point of $\mathcal{L}_X$ if and only if its components pass two tests, (i) and (ii):*

(i) *When expanded as meromorphic functions with poles $q \neq 0, \infty$ only at roots of unity, $f^{(i)} \in \mathcal{L}$, i.e. represent values of the big J-function $\mathcal{J}_{pt}$ in permutation-equivariant theory of the point target space;*

(ii) *Away from $q = 0, \infty$, and roots of unity, $f^{i)}$ may have at most simple poles at $q = (\Lambda_j/\Lambda_i)^{1/m}$, $j \neq i$, $m = 1, 2, \ldots$, with the residues satisfying the recursion relations*

$$\mathrm{Res}_{q=(\Lambda_j/\Lambda_i)^{1/m}}\, f^{(i)}(q)\frac{dq}{q} = -\frac{Q^m}{C_{ij}(m)}\, f^{(j)}((\Lambda_j/\Lambda_i)^{1/m}),$$

where $C_{ij}(m)$ are explicitly described rational functions.

We even verified that the hypergeometric series

$$J^{(i)} = (1-q)\sum_{d\geq 0} \frac{Q^d}{\left(\prod_{r=1}^d (1-q^r)\right)\prod_{j\neq i}\prod_{r=1}^d (1-q^r\Lambda_i/\Lambda_j)}$$

pass test (ii). Now we are ready for test (i). Indeed, we know from Part I (or from Part III) that

$$(1-q)\Gamma_q(Q) := (1-q)e^{\sum_{k>0} Q^k/k(1-q^k)} = (1-q)\sum_{d\geq 0}\frac{Q^d}{\prod_{r=1}^d (1-q^r)}$$

lies in $\mathcal{L}$. According to Lemma,

$$J^{(i)} = \prod_{j\neq i} \frac{\Gamma_{q^{-1}}(\Lambda_i\Lambda_j^{-1}q^{Q\partial_Q})}{\Gamma_{q^{-1}}(\Lambda_j\Lambda_j^{-1})}\,(1-q)\Gamma_q(Q)$$

also lies in $\mathcal{L}$. Thus, we obtain

`Corollary 1.` *The $K^0(\mathbb{C}P^N)$-valued function*

$$J_{\mathbb{C}P^N} := \sum_{i=0}^N J^{(i)}\psi_i = (1-q)\sum_{d\geq 0}\frac{Q^d}{\prod_{j=0}^N\prod_{r=1}^d (1-P\Lambda_j^{-1}q^r)},$$

where $P = \mathcal{O}(-1)$ satisfies $\prod_{j=0}^N (1-P\Lambda_j^{-1}) = 0$, represents a value of of the big J-function $\mathcal{J}_{\mathbb{C}P^N}$.

`Remark.` Note that all summands with $d > 0$ are reduced rational functions of q, and so the Laurent polynomial part of $J_{\mathbb{C}P^N}$ consists of the dilaton shift term $1-q$ only. This means that $J_{\mathbb{C}P^N}$ represents the value of the big J-function $\mathcal{J}_{\mathbb{C}P^N}(\mathbf{t})$ at the input $\mathbf{t}=0$. Hence it is the small J-function (not only in permutation-equivariant but also) in the *ordinary* quantum K-theory of $\mathbb{C}P^N$. In this capacity it was computed in [4] by *ad hoc* methods.

One can derive this way many other applications. To begin with, consider quantum K-theory on the target E which is the total space of a vector bundle $E \to X$. To make the theory formally well-defined, one equips E with the fiberwise scaling action of a circle, T', and defines correlators by localization to fixed points $E^{T'} = X$ (the zero section of E). This results in systematic *twisting* of virtual structure sheaves on the moduli spaces $X_{g,n,d}$ as follows:

$$\mathcal{O}^{virt}_{g,n,d}(E) := \frac{\mathcal{O}^{virt}_{g,n,d}(X)}{\operatorname{Euler}^K_{T'}(E_{g,n,d})}, \quad E_{g,n,d} = (\mathrm{ft}_{n+1})_* \operatorname{ev}^*_{n+1}(E),$$

where the T'-equivariant K-theoretic Euler class of a bundle V is defined by

$$\operatorname{Euler}^K_{T'}(V) := \operatorname{tr}_{\lambda \in T'} \left(\sum_k (-1)^k \bigwedge^k V^* \right).$$

The division is possible in the sense that the T'-equivariant Euler class is invertible over the field of fractions of the group ring of T'. The elements $E_{g,n,d} \in K^0(X_{g,n,d})$ are invariant under permutations of the marked points. (In fact [2, 3], for $d \neq 0$, $E_{g,n,d} = \mathrm{ft}^* E_{g,0,d}$ where $\mathrm{ft} : X_{g,n,d} \to X_{g,n,d}$ forgets all marked points.) Thus, we obtain a well-defined permutation-equivariant quantum K-theory of E.

`Corollary 2.` *Let $X = \mathbb{C}P^N$, and $E = \oplus_{j=1}^M \mathcal{O}(-l_j)$. Then the following q-hypergeometric series*

$$I_E := (1-q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^N \prod_{r=1}^d (1 - P\Lambda_j^{-1} q^r)} \prod_{j=1}^M \frac{\prod_{r=-\infty}^{l_j d - 1} (1 - \lambda P^{-l_j} q^{-r})}{\prod_{r=-\infty}^{-1} (1 - \lambda P^{-l_j} q^{-r})}$$

represents a value of the big J-function in the permutation-equivariant quantum K-theory of E.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on the fibers of E as multiplication by λ^{-1}. The K-theoretic Poincaré pairing on X is twisted into $(a,b)_E = \chi(X; ab/\operatorname{Euler}^K_T(E))$.

`Example.` Let $X = \mathbb{C}P^1$, $E = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In I_E, pass to the non-equivariant limit $\Lambda_0 = \Lambda_1 = 1$:

$$\begin{aligned} I_E = &(1-q) + (1-\lambda P^{-1})^2 \times \\ &(1-q)\sum_{d>0} Q^d \frac{(1-\lambda P^{-1}q^{-1})^2 \cdots (1-\lambda P^{-1}q^{1-d})^2}{(1-Pq)^2(1-Pq^2)^2\cdots(1-Pq^d)^2}. \end{aligned}$$

The factor $(1-\lambda P^{-1})^2$, equal to $\mathrm{Euler}^K_{T'}$, reflects the fact that the part with $d > 0$ is a push-forward from $\mathbb{C}P^1$ to E. In the second non-equivariant limit, $\lambda = 1$, it would turn into 0 (since $(1-P^{-1})^2 = 0$ in $K^0(\mathbb{C}P^1)$. However, what the part with $d > 0$ is push-forward of, survives in this limit:

$$(1-q)\sum_{d>0} \frac{Q^d}{P^{2d-2}q^{d(d-1)}(1-Pq^d)^2}, \text{ where } (1-P)^2 = 0.$$

This example is usually used to extract information about "local" contributions of a rational curve $\mathbb{C}P^{-1}$ lying in a Calabi-Yau 3-fold with the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Note that decomposing the terms of this series into two summands: with poles at roots of unity, and with poles at 0 or ∞, we obtain non-zero Laurent polynomials in each degree d. They form the input $\mathbf{t} = \sum_{d>0} \mathbf{t}_d(q, q^{-1})Q^d$ of the big J-function whose value $\mathcal{J}_E(\mathbf{t})$ is given by the series.

Finally, note that though the input is non-trivial, it is defined over the λ-algebra $\mathbb{Q}[[Q]]$. This means that, although we are talking about permutation-equivariant quantum K-theory, the hypergeometric functions here, and in Corollary 2 in general, represent *symmetrized* K-theoretic GW-invariant, i.e. S_n-invariant part of the sheaf cohomology.

Similarly, one can introduce K-theoretic GW-invariants of the *super-bundle* ΠE (which is obtained from $E \to X$ by the "parity change" Π of the fibers) by redefining the virtual structure sheaves as

$$\mathcal{O}^{virt}_{g,n,d}(\Pi E) := \mathcal{O}^{virt}_{g,n,d}(X)\ \mathrm{Euler}^K_{T'}(E_{g,n,d}).$$

When genus-0 correlators of this theory have non-equivariant limits (e.g. when E is a positive line bundle, and $d > 0$), the limits coincide with the appropriate correlators of the submanifold $Y \subset X$ given by a holomorpfic section of ΠE.

`Corollary 3.` *Let $X = \mathbb{C}P^N$, and $E = \oplus_{j=1}^M \mathcal{O}(l_j)$. Then the following q-hypergeometric series*

$$I_{\Pi E} := (1-q)\sum_{d\geq 0} \frac{Q^d}{\prod_{j=0}^N \prod_{r=1}^d (1-P\Lambda_j^{-1}q^r)} \prod_{j=1}^M \frac{\prod_{r=-\infty}^{l_j d}(1-\lambda P^{l_j}q^r)}{\prod_{r=-\infty}^{0}(1-\lambda P^{l_j}q^r)}$$

represents a value of the big J-function in the permutation-equivariant quantum K-theory of E.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on fibers of E as multiplication by λ. The Poincaré pairing is twisted into $(a,b)_{\Pi E} = \chi(X; ab\,\mathrm{Euler}_T^K(E))$.

`Example.` When all $l_j > 0$, it is safe pass to the non-equivariant limit $\Lambda_j = 1$ and $\lambda = 1$:

$$I_{\Pi E} = (1-q)\sum_{q\geq 0} Q^d \frac{\prod_{j=1}^M \prod_{r=1}^{l_j d}(1-P^{l_j}q^r)}{\prod_{r=1}^d (1-Pq^r)^{N+1}},$$

which represents a value of the big J-function of $Y \subset \mathbb{C}P^N$, pushed-forward from $K^0(Y)$ to $K^0(\mathbb{C}P^N)$. Here Y is a codimension-M complete intersection given by equations of degrees l_j. Taking in account the degeneration of the Euler class in this limit, one may assume that P satisfies the relation $(1-P)^{N+1-M} = 0$.

When $\sum_j l_j^2 \leq N+1$, the Laurent polynomial part of this series is $1-q$, i.e. the corresponding input $\mathbf{t}$ of the J-function vanishes. In this case the series represents the small J-function of the ordinary quantum K-theory on Y. This result was obtained in [5] in a different way: based on the adelic characterization of the whole theory, but without the use of fixed point localization. As we have seen here, when $\mathbf{t} \neq 0$, the series still represents the value $\mathcal{J}_Y(\mathbf{t})$ in the *symmetrized* quantum K-theory of Y.

In Part V these results will be carried over to all toric manifolds X, toric bundles $E \to X$, or toric super-bundles ΠE. In fact, the intention to find a home for toric q-hypergeometric functions with non-zero Laurent polynomial part was one of the motivations for developing the permutation-equivariant version of quantum K-theory.

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