PERMUTATION-EQUIVARIANT QUANTUM K-THEORY IV.
$\mathcal{D}_q$-MODULES

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ABSTRACT. In Part II, we saw how permutation-equivariant quantum K-theory of a manifold with isolated fixed points of a torus action can be reduced via fixed point localization to permutation-equivariant quantum K-theory of the point. In Part III, we gave a complete description of permutation-equivariant quantum K-theory of the point by means of adelic characterization. Here we apply the adelic characterization to introduce the action on this theory of a certain group of $q$-difference operators. This action enables us to prove that toric $q$-hypergeometric functions represent K-theoretic GW-invariants of toric manifolds.

OVERRULED CONES AND $\mathcal{D}_q$-MODULES

In Part III, we gave the following adelic characterization of the big J-function $\mathcal{J}_{pt}$ of the point target space. In the space $\mathcal{K}$ of “rational functions” of $q$ (consisting in fact of series in auxiliary variables with coefficients which are rational functions of $q$), let $\mathcal{L}$ denote the range of $\mathcal{J}_{pt}$. We showed that an element $f \in \mathcal{K}$ lies in $\mathcal{L}$ if and only if Laurent series expansions $f(\zeta)$ of $f$ near $q = \zeta^{-1}$ satisfy

(i) $f(1) = (1 - q)e^{\tau/(1 - q)} \times (\text{power series in } q - 1)$ for some $\tau \in \Lambda_+$,

(ii) when $\zeta \neq 1$ is a primitive $m$-th root of unity,

$$f(\zeta)(q^{1/m}/\zeta) = \Psi^m(f(1)/(1 - q)) \times (\text{power series in } q - 1),$$

where $\Psi^m$ is the Adams operation extended from $\Lambda$ by $\Psi^m(q) = q^m$;

(iii) when $\zeta \neq 0, \infty$ is not a root of unity, $f(\zeta)(q/\zeta)$ is a power series in $q - 1$.

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1For convergence purposes, we assume that the Adams operations $\Psi^k$ on $\Lambda$ with $k > 1$ increase certain filtration $\Lambda \supset \Lambda_+ \supset \Lambda_{+++} \supset \cdots$, and that the domain of the J-function is $\Lambda_+$. 

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Another way to phrase (i) is to say that $f_{(1)}$ lies in the range $L_{\text{fake}}$ of the ordinary (or fake) J-function $J_{\text{ord}}^{\text{pt}}$ in the space $\hat{K} = \Lambda((q - 1))$ of Laurent series in $q - 1$:

$$L_{\text{fake}} = \bigcup_{\tau \in \Lambda_+} (1 - q)e^{\tau/(1 - q)}\hat{K}_+, \quad \hat{K}_+ := \Lambda[[q - 1]].$$

The range $L_{\text{fake}}$ is an example of an overruled cone: Its tangent spaces $T_{\tau} = e^{\tau/(1 - q)}\hat{K}_+$ are tangent to $L_{\text{fake}}$ along the subspaces $(1 - q)T_{\tau}$ (which sweep $L_{\text{fake}}$ as the parameter $\tau$ varies through $\Lambda_+$.\footnote{In terminology of S. Barannikov [1], this is a variation of semi-infinite Hodge structures: The flags $\cdots \subset (1 - q)T_{\tau} \subset T_{\tau} \subset (1 - q)^{-1}T_{\tau} \subset \cdots$ vary in compliance with “Griffiths’ transversality condition”.
}) As it will be explained shortly, this property leads to the invariance of $L_{\text{fake}}$ to certain finite-difference operators.

Recall that in permutation-equivariant quantum K-theory, we work over a $\lambda$-algebra, a ring equipped with Adams homomorphisms $\Psi^m, m = 1, 2, \ldots, \Psi^1 = \text{Id}, \Psi^m \Psi^l = \Psi^{ml}$. Let us take $\Lambda := \Lambda_0[[\lambda, Q]]$ with $\Psi^m(\lambda) = \lambda^m, \Psi^m(Q) = Q^m$, where $\Lambda_0$ is any ground $\lambda$-algebra over $\mathbb{C}$.

Consider the algebra of finite-difference operators in $Q$. Such an operator is a non-commutative expression $D(Q, 1 - qQ\partial Q, q^{\pm 1})$. Clearly, the space $\hat{K}_+ = \Lambda[[q - 1]]$ (as well as $(1 - q)\hat{K}_+$) is a $D_q$-module. Consequently each ruling space $(1 - q)T_{\tau} = e^{\tau/(1 - q)(1 - q)\hat{K}_+}$ is a $D_q$-module too. Indeed,

$$qQ\partial Q e^{\tau(Q)/(1 - q)} = e^{\tau(Q)/(1 - q)}Q e^{(\tau(qQ) - \tau(Q))/(1 - q)},$$

where the second factor lies in $\hat{K}_+$. Moreover, we have

**Proposition.** $e^{\lambda D(Q, 1 - qQ\partial Q, q)/(1 - q)} L_{\text{fake}} = L_{\text{fake}}$.

**Proof.** The ruling space $(1 - q)T_{\tau}$ is a $D_q$-module, and hence invariant under $D$. Therefore for $f \in (1 - q)T_{\tau}$, we have $Df/(1 - q) \in T_{\tau}$, i.e. the vector field defining the flow $t \mapsto e^{t\lambda D/(1 - q)}$ is tangent to $L_{\text{fake}}$, and so the flow preserves $L_{\text{fake}}$. It remains to take $t = 1$, which is possible thanks to $\lambda$-adic convergence.

**Remark.** Generally speaking, linear transformation $e^{\lambda D/(1 - q)}$ does not preserve ruling spaces $(1 - q)T_{\tau}$, but transforms each of them into another such space. Indeed, preserving $L_{\text{fake}}$, it transform tangent spaces $T_{\tau}$ into tangent spaces, and since it commutes with multiplication by $1 - q$, it also transforms ruling spaces $(1 - q)T_{\tau}$ into ruling spaces.
Likewise, cone $L \subset K$ is ruled by subspaces comparable to $(1 - q)K_+$, namely by $(1 - q)L_\tau$, where $L_\tau := e^{\sum_{k>0} \Psi^k(\tau) / k(1-q^k)}K_+$. However $L_\tau$ are not tangent to $L$. Nonetheless the following result holds.

**Theorem.** The range $L$ of the big $J$-function $J_{pt}$ in the permutation-equivariant quantum K-theory of the point target space is preserved by operators of the form
e^{\sum_{k>0} \lambda^k \Psi^k(D(1-q^kQ\partial Q,q^{\pm 1}))/k(1-q^k)}.

**Remarks.**
(1) The operator $D$ has constant coefficients, i.e. is independent of $Q$.
(2) Note that $\Psi^k(q^kQ\partial Q) = q^kQ\partial Q$, and not $q^kQ\partial Q$ as in the exponent.
(3) The reader is invited to check that the theorem and its proof are extended without any changes to the case finite difference operators in several variables $Q_1, \ldots, Q_K$. We will use the theorem in this more general form in Part V.

**Proof.** Assuming that $(1 - q)f \in L$, we use the adelic characterization of $L$ to show that $(1 - q)g \in L$, where
\[ g(q) := e^{\sum_{k>0} \lambda^k \Psi^k(D(1-q^kQ\partial Q,q^{\pm 1}))/k(1-q^k)}f(q). \]
First, this relationship between $g$ and $f$ also holds between $g(\zeta)$ and $f(\zeta)$ where however both sides need to be understood as Laurent series in $q - 1$. Since $f(\zeta) \in L^{\text{fake}}$, Proposition implies that $g(\zeta) \in L^{\text{fake}}$ too.

Next, applying $\Psi^m$ to both sides, we find
\[ \Psi^m(g(\zeta)) = e^{\sum_{l>0} \lambda^l \Psi^l(D(1-q^lQ\partial Q,q^{\pm 1}))/l(1-q^l)}\Psi^m(f(\zeta)). \]
On the other hand, for an $m$-th primitive root of unity $\zeta$, taking into account that $\Psi^m(q) = q^m$ turns after the change $q \mapsto q^{1/m}/\zeta$ into $q^l$, and that $q^mQ\partial Q$ turns after this change into $q^lQ\partial Q$, we find
\[ g(\zeta)(q^{1/m}/\zeta) = e^{\Delta} e^{\sum_{l>0} \lambda^l \Psi^l(D(1-q^lQ\partial Q,q^{\pm 1/m}))/l(1-q^l)}f(\zeta)(q^{1/m}/\zeta), \]
where the finite-difference operator $\Delta$ has coefficients regular at $q = 1$. Here we factor off the terms regular at $q = 1$ using the fact that our operators have constant coefficients, and hence commute. Namely, $e^{A+B/(1-q)}$, where $A$ and $B$ are regular at $q = 1$, can be rewritten as $e^{A+B/(1-q)}$.

We are given that $f(\zeta)(q^{1/m}/\zeta) = p \Psi^m(f(\zeta))$ where $p \in \hat{K}_+$. Since $[q^kQ\partial Q,Q] = (q - 1)Q^kQ\partial Q$ is divisible by $q - 1$, for any finite-difference
Comparing this expression with $\Psi_m$ at operator $\partial_4$. GIVENTAL

\[ q \text{ multiplication by } q \text{ for symbols of } \]

\[ \text{The result follows now from the theorem of the previous section.} \]

\[ \text{where } q \text{ is regular at } D_4 \text{ for some } \tau = 1, \text{ we conclude that } \]

\[ q \text{ is regular at } D_4 \text{ and hence } \]

\[ \text{We use } \]

\[ \text{Proof. We use } q\text{-Gamma-function } \]

\[ \Gamma_q(x) := e^{\sum_{r>0} x^r/k(1-q^k)} \sim \prod_{r=0}^{\infty} \frac{1}{1-xq^r} \]

for symbols of $q$-difference operators:

\[ \frac{\Gamma_q^{-1}(\lambda q^{-lQ\partial_2})}{\Gamma_q^{-1}(\lambda)} Q^d = Q^d \prod_{r=-\infty}^{0} (1 - \lambda q^r) = Q^d \prod_{r=0}^{ld-1} (1 - \lambda q^r), \]

\[ \frac{\Gamma_q^{-1}(\lambda q^{lQ\partial_2})}{\Gamma_q^{-1}(\lambda)} Q^d = Q^d \prod_{r=-\infty}^{0} (1 - \lambda q^r) = \frac{Q^d}{\prod_{r=1}^{ld} (1 - \lambda q^r)}, \text{ and } \]

\[ \frac{\Gamma_q^{-1}(\lambda)}{\Gamma_q^{-1}(\lambda q^{lQ\partial_2})} Q^d = Q^d \prod_{r=1}^{ld} (1 - \lambda q^r) \]

The result follows now from the theorem of the previous section. □
Application to fixed point localization

In Part II, we used fixed point localization to characterize the range (denote it \( L_X \)) of the big J-function in permutation- (and torus-) equivariant quantum K-theory of \( X = \mathbb{C}P^N \). Namely a vector-valued “rational function” \( f(q) = \sum_{i=0}^{N} f^{(i)}(q) \phi_i \) represents a point of \( L_X \) if and only if its components pass two tests, (i) and (ii):

(i) When expanded as meromorphic functions with poles \( q \neq 0, \infty \) only at roots of unity, \( f^{(i)} \in \mathcal{L} \), i.e. represent values of the big J-function \( J_{pt} \) in permutation-equivariant theory of the point target space;

(ii) Away from \( q = 0, \infty \), and roots of unity, \( f^{(i)} \) may have at most simple poles at \( q = (\Lambda_j/\Lambda_i)^{1/m} \), \( j \neq i, m = 1, 2, \ldots \), with the residues satisfying the recursion relations

\[
\text{Res}_{q=(\Lambda_j/\Lambda_i)^{1/m}} f^{(i)}(q) \frac{dq}{q} = -\frac{Q^m}{C_{ij}(m)} f^{(j)}((\Lambda_j/\Lambda_i)^{1/m}),
\]

where \( C_{ij}(m) \) are explicitly described rational functions.

We even verified that the hypergeometric series

\[
J^{(i)} = (1-q) \sum_{d \geq 0} \left( \prod_{r=1}^{d} (1-q^r) \right) \prod_{j \neq i} \prod_{r=1}^{d} (1-q^r \Lambda_i/\Lambda_j)
\]

pass test (ii). Now we are ready for test (i). Indeed, we know from Part I (or from Part III) that

\[
(1-q) \Gamma_q(Q) := (1-q) e^{\sum_{k>0} Q^k/(1-q^k)} = (1-q) \sum_{d \geq 0} \prod_{r=1}^{d} \frac{Q^d}{1-q^r}
\]

lies in \( \mathcal{L} \). According to Lemma,

\[
J^{(i)} = \prod_{j \neq i} \frac{\Gamma_{q^{-1}}(\Lambda_i \Lambda_j^{-1} q^Q \partial_q)}{\Gamma_{q^{-1}}(\Lambda_j \Lambda_i^{-1})} (1-q) \Gamma_q(Q)
\]

also lies in \( \mathcal{L} \). Thus, we obtain

**Corollary 1.** The \( K^0(\mathbb{C}P^N) \)-valued function

\[
J_{\mathbb{C}P^N} := \sum_{i=0}^{N} J^{(i)} \psi_i = (1-q) \sum_{d \geq 0} \prod_{j=0}^{d} \prod_{r=1}^{d} (1-P \Lambda_j^{-1} q^r),
\]

where \( P = \mathcal{O}(-1) \) satisfies \( \prod_{j=0}^{N} (1-P \Lambda_j^{-1}) = 0 \), represents a value of the big J-function \( J_{\mathbb{C}P^N} \).

**Remark.** Note that all summands with \( d > 0 \) are reduced rational functions of \( q \), and so the Laurent polynomial part of \( J_{\mathbb{C}P^N} \) consists of the dilaton shift term \( 1-q \) only. This means that \( J_{\mathbb{C}P^N} \) represents the
value of the big J-function $J_{\mathbb{C}P^N}(t)$ at the input $t = 0$. Hence it is the small J-function (not only in permutation-equivariant but also) in the ordinary quantum K-theory of $\mathbb{C}P^N$. In this capacity it was computed in [4] by ad hoc methods.

One can derive this way many other applications. To begin with, consider quantum K-theory on the target $E$ which is the total space of a vector bundle $E \to X$. To make the theory formally well-defined, one equips $E$ with the fiberwise scaling action of a circle, $T'$, and defines correlators by localization to fixed points $E^{T'} = X$ (the zero section of $E$). This results in systematic twisting of virtual structure sheaves on the moduli spaces $X_{g,n,d}$ as follows:

$$
O^\text{virt}_{g,n,d}(E) := \frac{O^\text{virt}_{g,n,d}(X)}{\text{Euler}_{T'}^K(E_{g,n,d})}, \quad E_{g,n,d} = (\text{ft}_{n+1})_* \text{ev}_{n+1}^*(E),
$$

where the $T'$-equivariant K-theoretic Euler class of a bundle $V$ is defined by

$$
\text{Euler}_{T'}^K(V) := \text{tr}_{\lambda \in T'} \left( \sum_k (-1)^k \bigwedge^k V^* \right).
$$

The division is possible in the sense that the $T'$-equivariant Euler class is invertible over the field of fractions of the group ring of $T'$. The elements $E_{g,n,d} \in K^0(X_{g,n,d})$ are invariant under permutations of the marked points. (In fact [2, 3], for $d \neq 0$, $E_{g,n,d} = \text{ft}^* E_{g,0,d}$ where $\text{ft} : X_{g,n,d} \to X_{g,n,d}$ forgets all marked points.) Thus, we obtain a well-defined permutation-equivariant quantum K-theory of $E$.

**Corollary 2.** Let $X = \mathbb{C}P^N$, and $E = \bigoplus_{j=1}^M O(-l_j)$. Then the following $q$-hypergeometric series

$$I_E := (1 - q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^N \prod_{r=1}^d (1 - PA_j^{-1} q^r)} \prod_{j=1}^M \prod_{r=-\infty}^{l_j-1} (1 - \lambda P^{-l_j} q^{-r})$$

represents a value of the big J-function in the permutation-equivariant quantum K-theory of $E$.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on the fibers of $E$ as multiplication by $\lambda^{-1}$. The K-theoretic Poincaré pairing on $X$ is twisted into $(a,b)_E = \chi(X; ab/ \text{Euler}_{T'}^K(E))$. 
Example. Let \( X = \mathbb{C}P^1 \), \( E = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). In \( I_E \), pass to the non-equivariant limit \( \Lambda_0 = \Lambda_1 = 1 \):

\[
I_E = (1 - q) + (1 - \lambda P^{-1})^2 \times (1 - q) \sum_{d > 0} Q^d \frac{(1 - \lambda P^{-1}q^{-1})^2 \cdots (1 - \lambda P^{-1}q^{-d})^2}{(1 - Pq)^2(1 - Pq^2)^2 \cdots (1 - Pq^d)^2}.
\]

The factor \((1 - \lambda P^{-1})^2\), equal to Euler\(K_T\), reflects the fact that the part with \( d > 0 \) is a push-forward from \( \mathbb{C}P^1 \) to \( E \). In the second non-equivariant limit, \( \lambda = 1 \), it would turn into 0 (since \((1 - P^{-1})^2 = 0\) in \( K^0(\mathbb{C}P^1) \)). However, what the part with \( d > 0 \) is push-forward of, survives in this limit:

\[
I_E = (1 - q) + (1 - q) \sum_{d > 0} Q^d \frac{P^{2d-2}q^{d(d-1)}(1 - Pq^d)}{(1 - Pq^d)^2}, \text{ where } (1 - P)^2 = 0.
\]

This example is usually used to extract information about “local” contributions of a rational curve \( \mathbb{C}P^1 \) lying in a Calabi-Yau 3-fold with the normal bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \).

Note that decomposing the terms of this series into two summands: with poles at roots of unity, and with poles at 0 or \( \infty \), we obtain non-zero Laurent polynomials in each degree \( d \). They form the input \( t = \sum_{d > 0} t_d(q, q^{-1})Q^d \) of the big J-function whose value \( j_{E}(t) \) is given by the series.

Finally, note that though the input is non-trivial, it is defined over the \( \lambda \)-algebra \( \mathbb{Q}[[Q]] \). This means that, although we are talking about permutation-equivariant quantum K-theory, the hypergeometric functions here, and in Corollary 2 in general, represent *symmetrized* K-theoretic GW-invariant, i.e. \( S_n \)-invariant part of the sheaf cohomology.

Similarly, one can introduce K-theoretic GW-invariants of the super-bundle \( \Pi E \) (which is obtained from \( E \to X \) by the “parity change” \( \Pi \) of the fibers) by redefining the virtual structure sheaves as

\[
\mathcal{O}_{g,n,d}^{\virt}(\Pi E) := \mathcal{O}_{g,n,d}^{\virt}(X) \text{ Euler}_{K}(E_{g,n,d}).
\]

When genus-0 correlators of this theory have non-equivariant limits (e.g. when \( E \) is a positive line bundle, and \( d > 0 \)), the limits coincide with the appropriate correlators of the submanifold \( Y \subset X \) given by a holomorphic section of \( \Pi E \).

**Corollary 3.** Let \( X = \mathbb{C}P^N \), and \( E = \oplus_{j=1}^M \mathcal{O}(l_j) \). Then the following \( q \)-hypergeometric series

\[
I_{\Pi E} := (1 - q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^N \prod_{r=1}^d (1 - P \Lambda_j^{-1} q^r)} \prod_{j=1}^M \prod_{r=-\infty}^{l_j d} (1 - \lambda P^{l_j q^r})
\]
represents a value of the big J-function in the permutation-equivariant quantum K-theory of $E$.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on fibers of $E$ as multiplication by $\lambda$. The Poincaré pairing is twisted into $(a, b)_{\Pi E} = \chi(X; ab \text{Euler}_T^K (E))$.

**Example.** When all $l_j > 0$, it is safe pass to the non-equivariant limit $\Lambda_j = 1$ and $\lambda = 1$:

$$I_{\Pi E} = (1 - q) \sum_{q \geq 0} Q^d \prod_{j=1}^M \prod_{r=1}^{l_j d} (1 - P^l q^r) \prod_{r=1}^{d} (1 - P^r q^{N+1})$$

which represents a value of the big J-function of $Y \subset \mathbb{C}P^N$, pushed-forward from $K^0(Y)$ to $K^0(\mathbb{C}P^N)$. Here $Y$ is a codimension-$M$ complete intersection given by equations of degrees $l_j$. Taking in account the degeneration of the Euler class in this limit, one may assume that $P$ satisfies the relation $(1 - P)^{N+1-M} = 0$.

When $\sum_j l_j^2 \leq N + 1$, the Laurent polynomial part of this series is $1 - q$, i.e. the corresponding input $t$ of the J-function vanishes. In this case the series represents the small J-function of the ordinary quantum K-theory on $Y$. This result was obtained in [5] in a different way: based on the adelic characterization of the whole theory, but without the use of fixed point localization. As we have seen here, when $t \neq 0$, the series still represents the value $J_Y(t)$ in the symmetrized quantum K-theory of $Y$.

In Part V these results will be carried over to all toric manifolds $X$, toric bundles $E \to X$, or toric super-bundles $\Pi E$. In fact, the intention to find a home for toric $q$-hypergeometric functions with non-zero Laurent polynomial part was one of the motivations for developing the permutation-equivariant version of quantum K-theory.

**References**


