PERMUTATION-EQUIVARIANT QUANTUM K-THEORY X. QUANTUM HIRZEBRUCH-RIEMANN-ROCH IN GENUS 0

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ABSTRACT. We extract genus 0 consequences of the all genera Quantum HRR formula proved in Part IX. This includes re-proving and generalizing the adelic characterization of genus 0 quantum K-theory found in [5]. Extending some results of Part VIII and of [5], we derive the invariance of a certain variety (the "big J-function"), constructed from the genus 0 descendant potential of permutation-equivariant quantum K-theory, under the action of certain finite difference operators in Novikov's variables, apply this to reconstructing the whole variety from one point on it, and give an explicit description of it in the case of the point target space.

1. ADELIC CHARACTERIZATION

Let $\mathcal{F}_X$ denote the genus-0 descendant potential of permutation-equivariant quantum K-theory on a compact Kähler manifold X, or more precisely, its *dilaton-shifted* version. By definition (see Part IX)

$$\mathcal{F}_X(\mathbf{v} + \mathbf{t}) := \sum_{\mathbf{l},d} \frac{Q^d}{\prod_r l_r!} \langle \ldots, \mathbf{t}_1, \ldots; \ldots; \ldots, \mathbf{t}_k, \ldots; \ldots \rangle_{0,\mathbf{l},d}.$$

Here $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_k, \ldots)$ is a sequence of Laurent polynomials in q with vector coefficients in $K := K^0(X) \otimes \Lambda$, $\mathbf{v} := (1-q)(\mathbf{1}, \mathbf{1}, \ldots)$ is the *dilaton vector*, and $\mathbf{1}$ is the unit in $K^0(X)$. The *ground ring* Λ, containing Novikov's variables Q, is a λ-algebra, i.e. is equipped with Adams' operations Ψ^m, $m = 1, 2, \ldots$, $\Psi^1 = \mathrm{id}$, which act on Novikov's variables by $\Psi^m(Q^d) = Q^{md}$. We assume that Ψ^m with $m > 1$ increase the descending filtration of Λ by the powers Λ_+^d of a certain ideal Λ_+ containing all Novikov's monomials Q^d with $d \neq 0$. The correlators $\langle \ldots \rangle_{g,\mathbf{l},d}$ are defined in terms of K-theory on the moduli

space $X_{g,n,d}$ of genus-g degree-d stable maps to X of compact complex nodal curves with n marked points. Let h denote the automorphism of $X_{g,n,d}$ induced by the renumbering of the n marked points with the cycle structure determined by a partition $\mathbf{l} = (l_1, \ldots, l_r, \ldots)$, where l_r stands for the number of cycles of length $r = 1, 2, \ldots$. The correlator

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_{l_1}; \mathbf{v}_1, \ldots, \mathbf{v}_{l_2}; \ldots; \mathbf{w}_1, \ldots, \mathbf{w}_{l_r}; \ldots \rangle_{g,\mathbf{l},d},$$

where $\mathbf{u}_i, \mathbf{v}_j, \mathbf{w}_k$ are K-valued Laurent polynomials of q, is defined as follows. To such a Laurent polynomial, say, $\mathbf{w}(q) = \sum_m \phi_m q^m$, $\phi_m \in K^0(X) \otimes \Lambda$, we associate an h-equivariant vector bundle W on $X_{g,n,d}$:

$$W = \bigotimes_{\alpha=1}^{r} \sum_m \mathrm{ev}_{\sigma_\alpha}^*(\phi_m) L_{\sigma_\alpha}^{\otimes m},$$

where $\sigma_1, \ldots, \sigma_r$ are indices of marked points cyclically permuted by h, $\mathrm{ev}_\sigma : X_{g,n,d} \to X$ is the evaluation map at the marked point σ, and L_σ is the line orbibundle over $X_{g,n,d}$ formed by the cotangent lines to the curves at the σth marked point. With this notation, the above correlator equals

$$\left(\prod_{r=1}^{\infty} \frac{1}{r^{l_r}} \right) \mathrm{str}_h \, H^* \left(X_{g,n,d}; \mathcal{O}_{g,n,d} \bigotimes_{i=1}^{l_1} U_i \bigotimes_{j=1}^{l_2} V_j \cdots \bigotimes_{k=1}^{l_r} W_k \cdots \right),$$

where $\mathcal{O}_{g,n,d}$ is the virtual structure sheaf introduced by Y.-P. Lee [6].

With the function $\mathcal{F}_X$ we associate the "big J-function" which to a sequence $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots)$ of K-valued Laurent polynomials in q associates the following sequence $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots)$ of K-valued rational functions[1] of q:

$$\mathbf{f}_1 = (1 - q)\mathbf{1} + \mathbf{t}_1(q) + \sum_{\mathbf{l},d,\alpha} \frac{Q^d \phi_\alpha}{\prod_s l_s!} \langle \frac{\phi^\alpha}{1 - qL}, \mathbf{t}_1(L), \ldots; \mathbf{t}_2(L), \ldots \rangle_{d,\mathbf{l}+\mathbf{1}_1}$$

$$\ldots$$

$$\mathbf{f}_r = (1 - q)\mathbf{1} + \mathbf{t}_r(q) + \sum_{\mathbf{l},d,\alpha} \frac{Q^d \phi_\alpha}{\prod_s l_s!} \langle \frac{\phi^\alpha}{1 - qL}, \mathbf{t}_k(L), \ldots; \mathbf{t}_{2k}(L), \ldots \rangle_{d,\mathbf{l}+\mathbf{1}_1}$$

$$\ldots$$

Here the cycle structure $\mathbf{l} + \mathbf{1}_1$ is obtained from $\mathbf{l}$ by adding one cycle of length 1.

The range of the J-function is a (formal, infinite dimensional) subvariety in the space $\mathcal{K}^\infty$ of sequenses $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots)$ of K-valued rational

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[1]We remind that by a rational function, Laurent polynomial, etc. in q we mean a Q-series whose coefficients are such rational functions, Laurent polynomials, etc.

functions in q allowed to have poles at $q = 0, \infty$, or roots of unity. We denote this range by $\mathcal{L}_X$.

Another way to describe $\mathcal{L}_X$ is to consider $\mathcal{F}_X$ as a family of functions of the first input $\mathbf{t}_1$ depending on $\mathbf{t}_2, \mathbf{t}_3, \ldots$ as parameters. Then the map $\mathbf{t}_1 \mapsto \mathbf{f}_1(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \ldots)$ parameterizes the graph of the differential $d_{t_1} \mathcal{F}_X$ for given values of the parameters $\mathbf{t}_2, \mathbf{t}_3, \ldots$, and $\mathbf{f}_r :=$ $\mathbf{f}_1(\mathbf{t}_r, \mathbf{t}_{2r}, \mathbf{t}_{3r}, \ldots)$. The sequences $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots)$ of differentials lie in the cotangent bundle of $T^* \mathcal{K}_+^\infty$ of the subspace $\mathcal{K}_+^\infty \subset \mathcal{K}^{(\infty)}$, consisting of sequences of K-valued Laurent polynomaials in q. The cotangent bundle space is identified with the whole of $\mathcal{K}^\infty$ by means of the polarization $\mathcal{K}^\infty = \mathcal{K}_+^\infty \oplus \mathcal{K}_-^\infty$, Lagrangian with respect to the symplectic form

$$\Omega^\infty(\mathbf{f}, \mathbf{g}) = \sum_{r=1}^\infty \frac{\Psi^r}{r} \Omega(\mathbf{f}_r, \mathbf{g}_r),$$

$$\Omega(f, g) = - \left[\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty} \right] (f(q^{-1}), g(q)) \frac{dq}{q}.$$

Here $(a, b) := \chi(X; a \otimes b) = \int_X \mathrm{td}(T_X) \, \mathrm{ch}(a) \, \mathrm{ch}(b)$ is the K-theoretic Poincaré pairing on $K^0(X)$. By definition, the negative space $\mathcal{K}_-^\infty$ of the polarization consists of sequences $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots)$ satisfying $\mathbf{f}_r(\infty) = 0$, $\mathbf{f}_r(0) \neq \infty$.

Remark. Note that the graph of each $\mathbf{f}_r$ for fixed values of the parameters $\mathbf{t}_{2r}, \mathbf{t}_{3r}, \ldots$ is a Lagrangian variety in $(\mathcal{K}, \Omega)$ (where $\mathcal{K}$ is the space of K-valued Laurent polynomials in q), and the whole family of such Lagrangian varieties for, say, $r = 1$ determines $\mathcal{L}_X$. However, a value of $\mathbf{f}_1$, though "knows" its input $\mathbf{t}_1$, does not "remember" the values of the parameters $\mathbf{t}_2, \mathbf{t}_2, \ldots$. One reason to introduce $\mathcal{L}_X$ is that from a point on it, the corresponding argument $\mathbf{t} \in \mathcal{K}_+^\infty$ is reconstructed by projection along $\mathcal{K}_-^\infty$. On the other hand, there is no reason for $\mathcal{L}_X$ to be Lagrangian in $(\mathcal{K}^\infty, \Omega^\infty)$.

We will give an *adelic characterization* of $\mathcal{L}_X$ in terms of *fake* genus-0 K-theoretic GW-invariant of X. The fake holomorphic Euler characteristic of a bundle V over $X_{g,n,d}$ is defined by the right hand side of the Hirzebruch-RR formula:

$$\chi^{fake}(X_{g,n,d}; V) := \int_{[X_{g,n,d}]} \mathrm{td}(T_{X_{g,n,d}}) \, \mathrm{ch}(V),$$

where $[X_{g,n,d}]$ is the virtual fundamental class, and $T_{X_{g,n,d}}$ is the virtual tangent bundle. The genus-0 descendant potential $\mathcal{F}_X^{fake}$ is defined as

a function of $t(q) = \sum_{m \geqslant 0} \phi_m (q-1)^m \in \mathcal{K}_+^{fake}$ by

$$\mathcal{F}_X^{fake}(t) = \sum_{n,d} \frac{Q^d}{n!} \langle t(L), \ldots, t(L) \rangle_{0,n,d}^{fake},$$

where $\langle \phi_{\alpha_1} L^{m_1}, \ldots, \phi_{\alpha_n} L^{m_n} \rangle_{0,n,d} := \chi^{fake} \left(X_{0,n,d}; \otimes_{i=1}^n \mathrm{ev}_i^*(\phi_{\alpha_i}) L_i^{m_i} \right)$.

The graph of the differential $d\mathcal{F}_X^{fake}$ is identified with a Lagrangian variety $\mathcal{L}_X^{fake} \subset (\mathcal{K}^{fake}, \Omega^{fake})$. By $\mathcal{K}^{fake}$ we denote the space of K-valued Laurent series in $q-1$. It is equipped with the symplectic form

$$\Omega^{fake}(f,g) := \mathrm{Res}_{q=1}(f(q^{-1}), g(q)) \frac{dq}{q},$$

Lagrangian polarization $\mathcal{K}_\pm^{fake}$, where $\mathcal{K}_+^{fake}$ consists of power series, and $\mathcal{K}_-^{fake}$ consists of the principal parts of the Laurent series, as well as the dilaton vector $(1-q)\mathbf{1}$. Explicitly, $\mathcal{L}_X^{fake}$ consists of vector-valued Laurent series in $q-1$ of the form

$$1 - q + t(q) + \sum_{\alpha,n,d} \frac{Q^d \phi_\alpha}{n!} \langle \frac{\phi^\alpha}{1-qL}, t(L), \ldots, t(L) \rangle_{0,n+1,d}^{fake}.$$

In fact $\mathcal{L}_X^{fake}$ is an *overruled Lagrangian cone*, i.e. each tangent space T of $\mathcal{L}_X^{fake}$ contain $(1-q)T \subset \mathcal{L}_X^{fake}$ and is tangent to $\mathcal{L}_X^{fake}$ everywhere along $(1-q)T$. The cone $\mathcal{L}_X^{fake}$ can be explicitly described in terms of cohomological GW-theory of X (see [1, 3, 5] or Part IX).

Theorem 1. *Let* $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_r, \ldots)$ *be a sequence of rational functions of q with values in $K = K^0(X) \otimes \Lambda$ considered as an element of the symplectic loop space $(\mathcal{K}^\infty, \Omega^\infty)$. This sequence represents a point in the range $\mathcal{L}_X \subset \mathcal{K}^\infty$ of the "big J-function" of permutation-equivariant quantum K-theory of X if and only if it satisfies the following criteria:*

(i) For each $r = 1, 2, \ldots$, the Laurent series expansion $\mathbf{f}_r^{(1)}$ of $\mathbf{f}_r(q)$ near $q = 1$ lies in the Lagrangian submanifold $\mathcal{L}_X^{fake} \subset (\mathcal{K}^{fake}, \Omega^{fake})$ representing the graph of the differential of of the genus 0 descendant potential $\mathcal{F}_X^{fake}$ of fake quantum K-theory of X;

(ii) For each r, and each root of unity $\zeta \neq 1$, the Laurent series expansion $\mathbf{f}_r^{(\zeta)}$ of $\mathbf{f}_r(q^{1/m}/\zeta)$ near $q = 1$, where m is the primitive order of ζ, lies in the Lagrangian subspace

$$\triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes_{\Psi^m(\Lambda)} \Lambda,$$

where the operator of multiplication $\triangle_\zeta$ is given by the formula

$$\triangle_\zeta := e^{\sum_{k>0} \left(\frac{\Psi^k(T_X^*-1)}{k(1-\zeta^{-k}q^{k/m})} - \frac{\Psi^{km}(T_X^*-1)}{k(1-q^{km})} \right)}.$$

Theorem 1 generalizes several previously known results. Setting $\mathbf{t}_k = 0$ for all $k > 1$ we obtain the result of [5] which characterizes the version of genus-0 quantum K-theory which does not involve permutations of the marked points. Setting all $\mathbf{t}_k$ with $k > 1$ equal to each other, we obtain an adelic characterization of permutation-equivariant J-function studied in Part VII and Part VIII.

2. Example: $X = pt$

In this case, $\mathcal{K}^\infty$ consists of sequences $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ of scalar (i.e. Λ-valued) rational functions of q with poles at $q = 0, \infty$ or roots of unity, and $\mathcal{K}^\infty_+$ consists of sequences $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_r, \ldots)$ of scalar Laurent polynomials in q.

Theorem 2. *The range $\mathcal{L}_{pt} \subset \mathcal{K}^\infty$ of the "big J-function" consists of sequences $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots) \in \mathcal{K}^\infty$ of the form*

$$\mathbf{f}_1(q) = (1-q)e^{\sum_{k>0} \Psi^k(\tau_k)/k(1-q^k)} \, \mathbf{t}_1(q, q^{-1})$$

$$\cdots$$

$$\mathbf{f}_r(q) = (1-q)e^{\sum_{k>0} \Psi^k(\tau_{kr})/k(1-q^k)} \, \mathbf{t}_r(q, q^{-1})$$

$$\cdots$$

where $\mathbf{t}_k \in \mathcal{K}_+$ are scalar Laurent polynomials Λ_+-close to 1, and $\tau_k \in \Lambda_+$ are arbitrary scalar parameters.

Proof. We derive this by verifying criteria (i), (ii) of Theorem 1.

It is known (see, for instance, [2]) that the $\mathcal{L}^{fake}_{pt}$ consists of Laurent series of the form

$$(1-q)e^{\tau/(1-q)} \, t(q-1),$$

where t is a power series in $q-1$ which is Λ_+-close to 1, and $\tau \in \Lambda_+$ is an arbitrary scalar.

First, we expand $\mathbf{f}_r$ into Laurent series $\mathbf{f}_r^{(1)}$ in $q-1$:

$$\mathbf{f}_r^{(1)} = (1-q)e^{\mathcal{T}_r/(1-q)} \, t_r(q-1), \text{ where } \mathcal{T}_r = \sum_{k>0} \Psi^k(\tau_{kr})/k^2,$$

and t_r are some power series in $q-1$. We use here that $1/(1-q^k) = 1/k(1-q) + \mathcal{O}(1)$. Thus $\mathbf{f}_r^{(1)} \in \mathcal{L}^{fake}_{pt}$, and criterion (i) is fulfilled.

Note that the tangent space to $\mathcal{L}^{fake}_{pt}$ at the point $\mathbf{f}_r^{(1)}$ has the form $e^{\mathcal{T}_r/(1-q)}\mathcal{K}^{fake}_+$.

In the case $X = pt$, the operator $\triangle_\zeta$ consists in multiplication by an invertible power series in $q-1$. Therefore, for a primitive mth root of

unity ζ,

$$\triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_{pt}^{fake}) \otimes_{\Psi^m(\Lambda)} \Lambda = e^{\Psi^m[\mathcal{T}_{rm}/(1-q)]} \mathcal{K}_+^{fake}.$$

We want to check that this subspace contains the Laurent series expansion $\mathbf{f}_r^{(\zeta)}$ of $\mathbf{f}_r(q^{1/m}/\zeta)$ near $q = 1$. We have:

$$\mathbf{f}_r^{(\zeta)} = (1 - q^{1/m}/\zeta) e^{\sum_{k>0} \Psi^k(\tau_{kr})/k(1-q^{k/m}/\zeta^k)} \mathbf{t}_r(q^{1/m}/\zeta, q^{-1/m}\zeta).$$

The terms in the exponent which have pole at $q = 1$ come from the values of k divisible by m. For $k = lm$ we have:

$$\sum_{l>0} \frac{\Psi^{lm}(\tau_{lmr})}{lm(1-q^l)} = \Psi^m\left(\frac{\mathcal{T}_{rm}}{1-q}\right) + \text{terms regular at } q = 1.$$

Thus, criterion (ii) is also fulfilled.

Thus, the family of sequences $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ described in Theorem 2 lie in $\mathcal{L}_{pt}$. Note that at $\tau = 0$, this family contains the Λ_+-neighborhood of the dilaton vector in $\mathcal{K}_+^\infty$. At $\mathbf{t} = \mathbf{1}$, the projection of $\mathbf{f}$ to $\mathcal{K}_+^\infty$ along $\mathcal{K}_-^\infty$ is $[\mathbf{f}]_+ = (1 - q + \tau_1, \ldots, 1 - q + \tau_r, \ldots)$. Thus, the tangent space at the dilaton point to the entire domain of $\mathcal{F}_{pt}$ is covered by our family. It follows now from the formal Implicit Function Theorem, that the family parameterizes the whole of $\mathcal{L}_{pt}$.

3. $\mathcal{D}_q$-SYMMETRIES

We generalize here results of [5, 4], and Part VIII about symmetries of genus-0 quantum K-theory induced by finite-difference operators in Novikov's variables.

Pick an integer basis $p_1, \ldots, p_s$ in $H_2(X; \mathbb{Q})$ with respect to which all degrees of holomorphic curves in X are expressed by vectors $d = (d_1, \ldots, d_s)$ with non-negative components d_i. Let P_i be the line bundle over X whose 1st Chern class is $-p_i$. Let $\mathcal{D}_q$ denote the algebra of finite difference operators in Novikov's variables. By definition it consists of non-commutative polynomial expressions formed from multiplication operators Q_i, translation operators $q^{Q_i \partial_{Q_i}}$, and can have Laurent polynomials in q in the role of coefficients. We make $\mathcal{D}_q$ act on the space $\mathcal{K}$ of K-valued rational functions of q so that Q_i act naturally as multiplication by $Q_i \in \Lambda_+$, while the translation operators act on K-valued functions of Q by $P_i q^{Q_i \partial_{Q_i}}$, i.e. by

$$\mathcal{F}(Q_1, \ldots, Q_s) \mapsto P_i \mathcal{F}(\ldots, Q_{i-1}, qQ_i, Q_{i+1}, \ldots).$$

For convergence purposes we will further assume that our operators $D(Pq^{Q\partial_Q}, Q, q)$ have "small free terms", i.e. $D(1, 0, q) \in \Lambda_+[q, q^{-1}]$.

Theorem 3. *Let $D = (D_1, \ldots, D_k, \ldots)$ be a sequence of finite differ-ence operators $D_k(Pq^{Q\partial_Q}, Q, q)$. Then the following transformation on the space $\mathcal{K}^\infty$ of sequences $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ of vector-valued rational function preserves $\mathcal{L}_X$:*

$$\mathbf{f}_r \mapsto e^{\sum_{k>0} \Psi^k(D_{kr}(Pq^{kQ\partial_Q}, Q, q))/k(1-q^k)} \mathbf{f}_r, \quad r = 1, 2, \ldots.$$

Remark. Recall that Adams' operations Ψ^k act naturally on $K^0(X)$, act of functions of q by $\Psi^k(q) = q^k$, act through the λ-algebra struc-ture on the coefficient ring Λ, and in particular by $\Psi^k(Q^d) = Q^{kd}$ on Novikov's variables. Consequently, multiplication by P_i hidden in $\Psi^k(D(Pq^{Q\partial_Q}, Q, q))$ turns after the application of Ψ^k into multiplica-tion by P_i^k. However,

$$\Psi^k(q^{Q_i \partial_{Q_i}} Q^d) = \Psi^k(q^{d_i} Q^d) = q^{kd_i} Q^{kd} = q^{Q_i \partial_{Q_i}} \Psi^k(Q^d),$$

i.e. $\Psi^k(q^{Q\partial_Q}) = q^{Q\partial_Q}$. Therefore $\Psi^k(Pq^{kQ\partial_Q}) = (Pq^{Q\partial_Q})^k$. Thus, the operators in the exponent act as legitimate finite difference operators in our representation, i.e. as combinations of multiplications by Q_i and twisted translation operators $P_i q^{Q_i \partial_{Q_i}}$.

Proof. Suppose $\mathbf{f} \in \mathcal{L}_X$. Then: (i) expansion $\mathbf{f}_r^{(1)}$ of $\mathbf{f}_r(q)$ near $q = 1$ lie in $\mathcal{L}_X^{fake}$, and (ii) expansion $\mathbf{f}_r^{(\zeta)}$ of $\mathbf{f}_r(q^{1/m}/\zeta)$ near $q = 1$, where $\zeta \neq 1$ is a primitive mth root of unity, lie in $\triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes_{\Psi^m(\Lambda)} \Lambda$.

It is known (see, for instance, [1, 5], or Section 3 in Part IX), that $\mathcal{L}_X^{fake}$ is an overruled Lagrangian cone in the symplectic loop space $\mathcal{K}^{fake}$, and that it is obtained by a certain loop group transformation $\triangle : \mathcal{H} \to \mathcal{K}^{fake}$ from the overruled Lagrangian cone $\mathcal{L}_X^H$ of quan-tum cohomology theory on X. Recall that $\mathcal{L}_X^H$ lies in the appropriate symplectic loop space $\mathcal{H} = H^{even}(X; \Lambda((z)))$, where $z = \log q$, and represents the graph of the differential of the genus 0 descendant po-tential of cohomological GW-theory of X. It follows (see [4]) from the divisor equations of quantum cohomology theory, that $\mathcal{L}_X^H$ is invariant under the action of operators $f \mapsto e^{D/z} f$ where $D(zQ\partial_Q - p, Q, z)$ is any (pseudo) differential operator (with "small free term") in Novikov's variables. Consequently, since the loop group transformation $\triangle$ does not depend on Q, the cone $\mathcal{L}_X^{fake}$ is similarly invariant under the action of operators $f \mapsto e^{D/(1-q)} f$, where $D((\log q)Q\partial_Q - p, Q, q - 1)$ is any differential operator. [2] The operator in the exponent here is required to have at most the 1st order pole at $q = 1$. Besides, the tangent spaces

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[2]Note that $\log q = \log(1 - (1 - q)) = -\sum_{k>0} (1-q)^k/k$ is a legitimate coefficient in $\mathcal{K}^{fake}$ (though not in $\mathcal{K}$), and $p_i = -\log P_i$ is well-defined under the Chern character identification of $K^0(X) \otimes \mathbb{Q}$ with $H^{even}(X; \mathbb{Q})$.

$T_f \mathcal{L}_X^{fake}$ and the *ruling spaces* $(1-q)T_f \mathcal{L}_X^{fake} \subset \mathcal{L}_X^{fake}$ are $\mathcal{D}$-modules, i.e. stay invariant under the operators D, and hence under the operators e^D (with no pole at $q = 1$). In particular, since $Pq^{Q\partial_Q} = e^{(\log q)Q\partial_Q - p}$, the invariance properties hold true for finite difference operators.

Now let $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_r, \ldots)$ be the result of the transformation described in the theorem and applied to $\mathbf{f}$. We intend to check that $\mathbf{g}$ satisfies the criteria (i), (ii) of Theorem 1. Since $1/(1 - q^k)$ has the 1st order pole at $q = 1$, it follows that $\mathbf{g}_r^{(1)} \in \mathcal{L}_X^{fake}$. Furthermore, according to the invariance properties described above, the tangent space to $\mathcal{L}_X^{fake}$ at $\mathbf{f}_{rm}^{(1)}$ is transformed to another tangent space (at $\mathbf{g}_{rm}^{(1)}$) determined by the polar part of the operator in the exponent. Therefore

$$T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake} = e^{\sum_{k>0} \Psi^k(D_{krm}(Pq^{kQ\partial_Q}, Q, q)/k^2(1 - q)} \, T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}.$$

On the other hand,

$$\mathbf{g}_r^{(\zeta)} = e^{\sum_{k>0} \Psi^k(D_{kr}(P(q^{1/m}/\zeta)^{kQ\partial_Q}, Q, q^{1/m}/\zeta))/k(1-q^{k/m}/\zeta^k)} \, \mathbf{f}_r(q^{1/m}/\zeta)$$

$$= e^{D(P(q^{1/m}/\zeta)^{Q\partial_Q}, Q, q-1)} \, e^{\sum_{l>0} \Psi^{lm}(D_{lmr}(Pq^{lQ\partial_Q}, Q, q))/l^2(1-q^m)} \, \mathbf{f}_r^{(\zeta)}.$$

The transition from the first to the second line uses a version of the Campbell-Hausdorff formula which rewrites $e^{A+B/(1-q)}$ as $e^C e^{B/(1-q)}$, where by A and B we denote operators without pole at $q = 1$. Since the commutator of $[A, B]$ is divisible by $q - 1$, the operator C also comes out without the pole.

Comparing with the formula for $T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake}$, we find that the factor $e^{B/(1-q)}$ coincides with

$$\Psi^m \left(e^{\sum_{k>0} \Psi^k(D_{krm}(Pq^{kQ\partial_Q}, Q, q))/k^2(1 - q)} \right).$$

Given that $\mathbf{f}_r^{(\zeta)} \in \triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes \Lambda$, and using the fact that $\triangle_\zeta$ does not depend on Q, we conclude that $e^{B/(1-q)} \mathbf{f}_r^{(\zeta)} \in \triangle_\zeta \Psi^m(T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes \Lambda$.

As we have discussed, the tangent space $T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake}$ is preserved by (differential and hence) finite difference operators. However, Adams' operation Ψ^m changes the representation by which such operators act. Namely, $\Psi^m((P_i q^{Q_i \partial_{Q_i}})^{1/m}) = P_i (q^{Q_i \partial_{Q_i}})^{1/m}$. Also note that $\zeta^{-Q_i \partial_{Q_i}}$, acting on Q^d by ζ^{-d_i}, acts trivially on $\Psi^m(Q^d) = Q^{md}$ when ζ is an mth root of unity. Therefore the operator $D(P(q^{1/m}/\zeta)^{Q\partial_Q}, Q, q - 1)$ (nicknamed as C in the Campbell-Hausdorff formula) acts in $\Psi^m(\mathcal{K}^{fake})$ as a legitimate pseudo-differential operator in this new representation, and preserves the subspace $\Psi^m(T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake})$. This implies that the same

is true for the subspace $\triangle_\zeta \Psi^m(T_{\mathbf{g}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes \Lambda$ in $\mathcal{K}^{fake}$, and shows that $\mathbf{g}_r^{(\zeta)} = e^C e^{B/(1-q)} \mathbf{f}_r^{(\zeta)}$ lies in this subspace.

Corollary (string flows). *The transformations*

$$\mathbf{f}_r \mapsto e^{\sum_{k>0} \Psi^k(\tau_{kr})/k(1-q^k)} \mathbf{f}_r, \ r = 1, 2, \ldots,$$

where $\tau_k \in \Lambda_+$, $k = 1, 2, \ldots$, preserve $\mathcal{L}_X$.

4. EXPLICIT RECONSTRUCTION

We generalize "explicit reconstruction" results from [4] and Part VIII. Let us begin with the following proposition.

Proposition ($\mathcal{D}_q$-module structure). *If $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ lies in $\mathcal{L}_X$, then $(D_1 \mathbf{f}_1, \ldots, D_r \mathbf{f}_r, \ldots)$, where $D_1, D_2, \ldots$ is any sequence of finite difference operators, also lies in $\mathcal{L}_X$.*

Proof. This is a simplified version of the previous arguments. We are given that $\mathbf{f}_r^{(\zeta)}$ pass the tests (i), (ii) of Theorem 1, i.e. $\mathbf{f}_r^{(1)} \in \mathcal{L}_X^{fake}$, and for primitive mth roots of unity $\zeta \neq 1$, $\mathbf{f}_r^{(\zeta)} \in \triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes \Lambda$. Then $\mathbf{f}_r^{(1)}$ lies in (its own) ruling space $(1-q) T_{\mathbf{f}_r^{(1)}} \subset \mathcal{L}_X^{fake}$, which is D_r-invariant. Therefore, for $\mathbf{g}_r := D_r \mathbf{f}_r$, we find that $\mathbf{g}_r^{(1)}$ lies in the same ruling space as $\mathbf{f}_r^{(1)}$, and the tangent spaces $T_{\mathbf{g}_r^{(1)}}$ and $T_{\mathbf{f}_r^{(1)}}$ coincide as well. Furthermore,

$$\mathbf{g}_r^{(\zeta)} = D_r(P(q^{1/m}/\zeta)^{Q\partial_Q}, Q, q^{1/m}/\zeta) \ \mathbf{f}_r^{(\zeta)} \in \triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}}) \otimes \Lambda,$$

since the operator here commutes with $\triangle_\zeta$ and preserves the subspace $\Psi^m(T_{\mathbf{f}_{rm}^{(1)}})$ in $\Psi^m(\mathcal{K}^{fake})$ (as was discussed in the proof of Theorem 3).

Remark. This Proposition together with the previous Corollary give another proof of Theorem 1. Namely, it is easy to check that $\mathbf{t} = \mathbf{0}$ is a critical point of $\mathcal{F}_{pt}$, which implies that the dilaton vector $(1-q)(\mathbf{1}, \mathbf{1}, \ldots)$ lies in $\mathcal{L}_{pt}$. Then, applying Theorem 4 with $D_r = \mathbf{t}_r(q, q^{-1})$ (multiplication by a Laurent polynomial), we conclude that $(1-q)\mathcal{K}_+^\infty$ lies in $\mathcal{L}_X$. Finally, applying the string flow from Corollary, we obtain the whole of $\mathcal{L}_{pt}$.

Theorem 4. *Suppose that the degree-2 classes $p_1, \ldots, p_s$ generate the cohomology algebra $H^{even}(X; \mathbb{Q})$, and hence (equivalently) the line bundles $P_1, \ldots, P_s$ multiplicatively generate $K^0(X) \otimes \mathbb{Q}$. Pick Laurent monomials P^{m_α} to form a linear basis in $K^0(X) \otimes \mathbb{Q}$. Let $c_{\alpha,r} \in \Lambda[q, q^{-}1]$, $r = 1, 2, \ldots$, be arbitrary[3] scalar Laurent polynomials in q,*

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[3]For consistency with our earlier understanding of the domain of $\mathcal{F}_X$ we should assume that $c_{\alpha,r}$ are Λ_+-close to 1.

and $\tau_{\alpha,k} \in \Lambda_+$, $k = 1, 2, \ldots$, be arbitrary "small" parameters. Suppose that $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ is a point in $\mathcal{L}_X$, and let $\mathbf{f}_r = \sum_d f_{r,d} Q^d$ be the Q-expansion of $\mathbf{f}_r$ (i.e. the coefficients $f_{r,d}$ do not depend on Q). With this notation, the following family $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_r, \ldots)$ of points in $\mathcal{K}^\infty$ lies in $\mathcal{L}_X$ and parameterizes the whole of it:

$$\mathbf{g}_r := \sum_d f_{r,d} Q^d e^{\sum_{k>0} \sum_\alpha \Psi^k(\tau_{\alpha,rk}) P^{km_\alpha} q^{k(m_\alpha,d)}/k(1-q^k)} \sum_\alpha c_{\alpha,r}(q) P^{m_\alpha} q^{(m_\alpha,d)}.$$

Here $(m,d) = \sum_i m^i d_i$ denotes the value of $\sum_i m^i p_i \in H^2(X)$ on $d \in H_2(X)$.

$\mathtt{Proof.}$ Let us assume first that $\tau_{\alpha,k}$ are free λ-algebra generators added to the ground ring Λ. We apply Theorem 3 and above Proposition over such an extended ground ring $\widetilde{\Lambda}$ to conclude that the following family $\widetilde{\mathbf{g}} = (\ldots, \widetilde{\mathbf{g}}_r, \ldots)$ of points lies in $\widetilde{\mathcal{L}}_X$:

$$\widetilde{\mathbf{g}}_r := \left(\sum_\alpha c_{\alpha,r}(Pq^{Q\partial_Q})^{m_\alpha} \right) e^{\sum_{k>0} \sum_\alpha \Psi^k(\tau_{\alpha,rk})(Pq^{Q\partial_Q})^{km_\alpha}/k(1-q^k)} \mathbf{f}_r.$$

Since $\tau_{\alpha,k}$ do not depend on Q, the operators in the exponent commute and can be computed explicitly on the monomials Q^d. Namely, $(q^{Q\partial_Q})^{km_\alpha} Q^d = q^{k(m_\alpha,d)} Q^d$. After that the operators on the left can be applied explicitly too. The resulting expression matches the one for $\mathbf{g}_r$ in the formulation of Theorem 4. Now we can invoke the change of the ground ring $\widetilde{\Lambda} \to \Lambda$ defined as the homomorphism of λ-algebras induced by the specialization of $\tau_{\alpha,k}$ to their values in Λ_+. It is easy to see that the descendant potential $\widetilde{\mathcal{F}}_X$ turns into $\mathcal{F}_X$ under this operation, and hence the variety $\widetilde{\mathcal{L}}_X$ turns into $\mathcal{L}_X$. Therefore the family $\mathbf{g}$ indeed lies in $\mathcal{L}_X$.

We can check now the same way as we did in the proof of Theorem 2 that the projection of this family to $\mathcal{K}_+^\infty$ along $\mathcal{K}_-^\infty$ covers the Λ_+-neighborhood of the dilaton vector. For instance, modulo Novikov's variables, taking all $\sum_\alpha c_{r,\alpha} P^{m_\alpha} = 1$, we have

$$g_{r,0} = f_{r,0} + \sum_\alpha \tau_{\alpha,r} P^{m_\alpha} + \text{terms with poles at } q \neq 0, \infty,$$

which gives a family[4] $[g_{r,0}]_+ = (1-q)\mathbf{1} + t$ with any q-independent value of t. In the other extreme, when all $\tau_{\alpha,k} = 0$, we have

$$\mathbf{g}_{r,0} = \left(\sum_\alpha c_{\alpha,r}(q)|_{Q=0} P^{m_\alpha} \right) f_{r,0},$$

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[4]We assume that $\mathcal{L}_X \ni \mathbf{f}$ lies in $\mathcal{K}^\infty$ in Λ_+-neighborhood of the dilaton vector, and in particular $f_{r,0} = (1-q)\mathbf{1} \mod \Lambda_+$.

which yields a family $[g_{r,0}]_+$ containing in projection along $(1-q)+t$ all multiples of $(1-q)$. Our claim follows from this by virtue of the formal Implicit Function Theorem.

5. Derivation of Theorem 1

One way to prove Theorem 1 would be to extend the arguments of [5] where a recursion relation for the J-function of quantum K-theory is derived from the virtual Kawasaki-RR formula on moduli spaces of genus 0 stable maps. Here we are rather interested in obtaining Theorem 1 as a consequence of the higher genus formula from Part IX for the total descendant potential $\mathcal{D}_X$ of permutation-equivariant quantum K-theory of X. However, we need to warn the reader, that such a derivation may in the end provide few advantages over the first, more direct method.

The genus 0 descendant potential $\mathcal{F}_X$ enters the total descendant potential $\mathcal{D}_X$ in the form

$$\mathcal{D}_X = e^{\sum_{k>0} \hbar^{-k} \Psi^k(\mathcal{F}_X(\mathbf{t}_k, \mathbf{t}_{2k}, \dots, \mathbf{t}_{rk}, \dots))/k} + \mathcal{O}(1),$$

where $\mathcal{O}(1)$ denotes all terms weighted by non-negative powers of $\hbar$, and containing contributions of higher genus curves. The adelic expression given for $\mathcal{D}_X$ in Part IX is built from the *adelic tensor product*

$$\underline{\mathcal{D}}_X(\{t_r^{(\zeta)}\}, \hbar, Q) := \bigotimes_{M=1}^{\infty} \mathcal{D}_{X/\mathbb{Z}_M}^{tw} \left(\left\{ t_{r(\zeta)}^{(\zeta)}/\sqrt{\hbar^{r(\zeta)}} \right\}_{\zeta:\, \zeta^M=1}, 1, Q^M \right),$$

where $\mathcal{D}_{X/\mathbb{Z}_M}^{tw}(\{t_{r(\zeta)}^{(\zeta)}\}, \hbar, Q)$ is the total descendant potential of a certain twisted fake quantum K-theory of the orbifold target space $X/\mathbb{Z}_M$. Recall that the inputs $t_{r(\zeta)}^{(\zeta)} \in \mathcal{K}_+^{fake} := K((q-1))$ of $\mathcal{D}_{X/\mathbb{Z}_M}^{tw}$ are labeled by primitive roots of unity ζ of *orders* $m(\zeta)$ dividing M, while the subscript $r(\zeta) := M/m(\zeta)$. This adelic product, subject to the suitable dilaton shift, is considered as a quantum state $\langle \underline{\mathcal{D}}_X \rangle$ in the quantization of the adelic product of symplectic loop spaces:

$$(\underline{\mathcal{K}}^\infty, \underline{\Omega}^\infty) := \prod_{r=1}^{\infty} (\underline{\mathcal{K}}^{(r)}, \underline{\Omega}^{(r)})$$

Likewise, the dilaton-shifted function $\mathcal{D}_X$ is considered as a quantum state $\langle \mathcal{D}_X \rangle$ in the quantization of the symplectic loop space $(\mathcal{K}^\infty, \Omega^\infty)$, where $\mathcal{K}^\infty$ consists of sequences $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_r, \dots)$ of K-valued rational functions of q.

The expression for $\langle \mathcal{D}_X \rangle$ in terms of $\langle \underline{\mathcal{D}}_X \rangle$ is induced by the symplectic *adelic map* $\_ : (\mathcal{K}^\infty, \Omega^\infty) \to (\underline{\mathcal{K}}^\infty, \underline{\Omega}^\infty)$. To a sequence $\mathbf{f}$ of

rational functions rational function of q, it associates a collection $\{\underline{\mathbf{f}}_r^{(\zeta)}\}$ of K-valued Laurent series in $(q-1)$, where $\underline{\mathbf{f}}_r^{(\zeta)}$ is the Laurent series expansion near $q=1$ of $\Psi^r(\mathbf{f}_r(q^{1/m(\zeta)}/\zeta))$. Note that $\underline{\mathbf{f}}_r^{(\zeta)}$ is related by Ψ^r to what we denoted earlier in this text by $\mathbf{f}_r^{(\zeta)}$.

The relation between the quantization spaces induced by the symplectic adelic map involves the change from the *standard* Lagrangian polarization $\underline{\mathcal{K}}_+^\infty$ (using which the adelic product $\otimes_M \mathcal{D}_{X/\mathbb{Z}_M}^{tw}$ is lifted to a function on $\underline{K}^\infty$) to the *uniform* polarization, in which the negative space coincides with adelic image of $\mathcal{K}_-^\infty \subset \mathcal{K}^\infty$. After the polarization's change, $\langle \mathcal{D}_X \rangle$ is identified with the restriction of $\langle \underline{\mathcal{D}}_X \rangle$ to the image of $\mathcal{K}_+^\infty = \mathcal{K}^\infty/\mathcal{K}_-^\infty$ in $\underline{\mathcal{K}}_+^\infty$ under the adelic map.

The whole expression has the structure of Wick's formula of summation over graphs, where the factors in $\otimes_M \langle \mathcal{D}_{X/\mathbb{Z}_M}^{tw} \rangle$ represent different types of vertices, and the edge propagators come from the change of polarization. The latter is equivalent to the application of $e^{\oplus_{r=1}^\infty \hbar^r \nabla_r/2}$, where ∇_r is a certain 2nd order differential operator acting only through the inputs $t_r^{(\zeta)}/\sqrt{\hbar^r}$ of $\underline{\mathcal{D}}_X^\infty$ of the same level r.

Our current goal is to extract from this expression a formula for the genus-0 part of $\log \mathcal{D}_X$, i.e. the terms with negative powers of $\hbar$:

$$\sum_{k>0} \hbar^{-k} \Psi^k(\mathcal{F}_0(\mathbf{t}_k, \mathbf{t}_{2k}, \ldots, \mathbf{t}_{rk}, \ldots))/k.$$

Recall that each $\mathcal{D}_{X/\mathbb{Z}_M}^{tw}$ has the form of the genus expansion $\mathcal{D}(\mathbf{t}) = e^{\sum_g \hbar^g \mathcal{F}_g(\mathbf{t})}$, where $\mathcal{F}_g$, which counts contributions of genus-0 stable maps to the orbifold $X/\mathbb{Z}_M$, is homogeneous of degree $2-2g$ in $\mathbf{t}$ *after* the dilaton shift. Contributions to genus-0 part of $\log \mathcal{D}_X$ come, however, from mappings of those connected orbicurves to $X/\mathbb{Z}_M$ whose ramified $\mathbb{Z}_M$-cover is rational (though not necessarily connected). If $\hat{\sigma}_i$, $i = 1, \ldots, n$, are marked points of ramification index $m_i = M/r_i$ on a connected orbicurve $\hat{\Sigma}$, then Hurwitz' formula for the Euler characteristic of the covering curve Σ gives

$$\mathrm{eu}(\Sigma) = M(2-2g-n) + \sum_{i=1}^n r_i = M\left(2-2g-\sum_i \left(1-\frac{1}{m_i}\right)\right).$$

For $\mathrm{eu}(\Sigma)$ to be positive, we must have $g=0$, all but at most 3 of m_i equal to 1, and those 3 (call them a, b, c) satisfy $1/a+1/b+1/c > 1$. Of course, this singles out the ADE-orbifold structures on $\mathbb{C}P^1$. However, it is easy to check that in the cyclic group $\mathbb{Z}_M$ there are no elements of orders a, b, c whose product is the identity, unless it is A_{m-1}-case: one of a, b, c equals 1, and the other 2 are equal to m. In other words, if $\Sigma \to \mathbb{C}P^1$ is a rational ramified $\mathbb{Z}_M$-cover, then base $\mathbb{C}P^1$ carries

2 ramification points of the same order m, Σ consists of $r = M/m$ disjoint copies of $\mathbb{C}P^1$, and the generator $h \in \mathbb{Z}_M$ acts on Σ by cyclically permuting the r copies in such a way that h^r acts on each of them as the multiplication by a primitive mth root of unity ζ at one of the ramification points (and hence by ζ^{-1} at the other).

We can now extract from the logarithm of the adelic product the vertex contribution into the genus-0 part of $\log \mathcal{D}_X$. It has the form:

$$\sum_{M=1}^{\infty} \left(\mathcal{F}_{X/\mathbb{Z}_M}^{tw} \left(\frac{t_M^{(1)}}{\sqrt{\hbar^M}} \right) + \frac{1}{2} \sum_{\substack{\zeta^M=1 \\ \zeta \neq 1}} \left\langle d^2 \mathcal{F}_{X/\mathbb{Z}_M}^{tw} \left(\frac{t_M^{(1)}}{\sqrt{\hbar^M}} \right) \frac{t_{r(\zeta)}^{(\zeta)}}{\sqrt{\hbar^{r(\zeta)}}}, \frac{t_{r(\zeta^{-1})}^{(\zeta^{-1})}}{\sqrt{\hbar^{r(\zeta^{-1})}}} \right\rangle \right).$$

In this expression we use a number of *ad-hoc* conventions. By $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$ we denote here the genus-0 descendant potential of fake twisted quantum K-theory of $X/\mathbb{Z}_M$. It is a function of $\{t_{r(\zeta)}^{(\zeta)}\}$, where ζ is an Mth root of unity of primitive order $m(\zeta)$, and $r(\zeta) = M/m(\zeta)$. The input $t_M^{(1)}$ corresponds to the unramified sector ($\zeta = 1$). In the left summand, we assume that the inputs from all other sectors are set to 0. In the previous discussion, this term represents contributions of totally unramified (and hence trivial) $\mathbb{Z}_M$-covers $\Sigma \to \mathbb{C}P^1$. The right sum represents covers ramified at 2 points. Each term comes from the quadratic differential of $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$, and is a bilinear form in $t_{r(\zeta)}^{(\zeta)}$ and $t_{r(\zeta^{-1})}^{(\zeta^{-1})}$. The coefficient matrix of the bilinear form consists, of course, of the 2nd derivatives of $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$ in the direction of the appropriate sectors (corresponding to ζ and ζ^{-1}), and evaluated at the input $t_M^{(1)}/\sqrt{\hbar^M}$ (as indicated in the formula), while *all inputs $t_{r(\eta)}^{(\eta)}$ from ramified sectors ($\eta^M = 1, \eta \neq 1$) are set to zero*. The linear operator $d^2 \mathcal{F}_{X/\mathbb{Z}_M}^{tw}(t_M^{(1)}/\sqrt{\hbar^M})$ acts from the space $(\mathcal{K}_r^{(\zeta)})_+ \ni t_r^{(\zeta)}$ to $(\mathcal{K}_r^{(\zeta)})_-$ which is identified by the symplectic pairing with the dual of $(\mathcal{K}_r^{(\zeta^{-1})})_+ \ni t_r^{(\zeta^{-1})}$.

Since the (dilaton-shifted) functions $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$ have homogeneity degree 2, we can rewrite the above sum by powers of $\hbar$:

$$\underline{\mathcal{F}} := \sum_{r=1}^{\infty} \hbar^{-r} \left(\mathcal{F}_{X/\mathbb{Z}_r}^{tw}(t_r^{(1)}) + \frac{1}{2} \sum_{\zeta \neq 1} \left\langle d^2 \mathcal{F}_{X/\mathbb{Z}_{m(\zeta)r}}^{tw}(t_{m(\zeta)r}^{(1)}) t_r^{(\zeta)}, t_r^{(\zeta^{-1})} \right\rangle \right).$$

There are two ways to approach our goal. One is to retain in $\log(e^{\oplus_r \hbar^r \nabla_r/2} e^{\underline{\mathcal{F}}})$ all terms of negative order in $\hbar$. This leads to summation over rooted trees. The other is to introduce the adelic version $\underline{\mathcal{L}}_X^{\infty} \subset (\underline{\mathcal{K}}^{\infty}, \underline{\Omega}^{\infty})$ of the variety $\mathcal{L}_X$ using $\underline{\mathcal{F}}_X$ and the standard polarization $\underline{\mathcal{K}}_{\pm}^{\infty}$.

Let us start with the first way. Put $\underline{\mathcal{F}} = \sum_r \hbar^{-r} \underline{\mathcal{F}}_r$, i.e. denote by $\underline{\mathcal{F}}_r$ the sum of the terms weighted by the rth power of $\hbar^{-1}$, and consider each $\underline{F}_r$ as a family of functions of $t_r = \{t_r^{(\zeta)}\}$ depending on $t_{mr}^{(1)}$ with $m > 1$ as parameters. Let us denote the parameters in $\underline{\mathcal{F}}_r$ by $t_{m,r}^{(1)}$ (thereby making the parameters with different r independent of each other). The differential operator ∇_r acts only through the variables t_r. Therefore

$$\log \left(e^{\hbar^r \nabla_r / 2} e^{\hbar^{-r} \underline{\mathcal{F}}_r} \right) = \hbar^{-r} \mathcal{F}_r + \mathcal{O}(\hbar^0),$$

where $\mathcal{F}_r$ is obtained by Wick's summation over trees from vertex contribution $\underline{\mathcal{F}}_r$ and edge propagator ∇_r. It still depends on $t_{m,r}^{(1)}$ with $m > 1$ as parameters.

Let us now employ our second approach, based on the families of Lagrangian varieties in adelic symplectic loop spaces.

We have

$$\underline{\mathcal{F}}_r = \mathcal{F}_{X/\mathbb{Z}_r}^{tw}(t_r^{(1)}) + \frac{1}{2} \sum_{\zeta \neq 1} \left\langle d^2 \mathcal{F}_{X/\mathbb{Z}_{m(\zeta)r}}^{tw}(t_{m(\zeta),r}^{(1)}) \, t_r^{(\zeta)}, t_r^{(\zeta^{-1})} \right\rangle,$$

where $t_{m,r}^{(1)}$ are viewed as parameters. Given their values, the graph of $d\underline{\mathcal{F}}_r$ is a Lagrangian submanifold $\underline{\mathcal{L}}_r$ in the adelic loop space $\underline{\mathcal{K}}^{(r)}$, which is the Cartesian product[5]

$$\underline{\mathcal{K}}^{(r)} := \prod_{\text{roots of unity } \zeta} \mathcal{K}_r^{(\zeta)},$$

where each $\mathcal{K}_r^{(\zeta)}$ is a copy of $\mathcal{K}^{fake} := K((q-1))$. It is equipped with the Cartesian product symplectic form

$$\underline{\Omega}^\infty(\mathbf{f}, \mathbf{g}) = \sum_\zeta \frac{1}{m(\zeta)} \sum_{r=1}^\infty \frac{1}{r} \operatorname{Res}_{q=1}(\underline{\mathbf{f}}_r^{(\zeta)}(q^{-1}), \underline{\mathbf{g}}_r^{(\zeta^{-1})}(q))^{(r)} \frac{dq}{q},$$

where the twisted Poincaré pairing $(\cdot, \cdot)^{(r)}$ on $K := K^0(X) \otimes \Lambda$ is characterized by

$$(\Psi^r a, \Psi^r b)^{(r)} = r \Psi^r(a, b).$$

Note that the symplectic form pairs the Lagrangian subspace $(\mathcal{K}_r^{(\zeta)})_\pm$ with $(\mathcal{K}_r^{(\zeta^{-1})})_\mp$ (here $+/-$ refer to "power series/principal part" respectively), and the standard polarization is $\underline{\mathcal{K}}_\pm^\infty := \prod_{\zeta,r} (\mathcal{K}_r^{(\zeta)})_\pm$.

Based on the explicit form of $\underline{\mathcal{F}}_r$, for each value of the parameters, we have $\underline{\mathcal{L}}_r = \prod_\zeta \mathcal{L}_r^{(\zeta)}$, where $\mathcal{L}_r^{(1)} \subset \mathcal{K}_r^{(1)}$ is the graph of $d\mathcal{F}_{X/\mathbb{Z}_r}^{tw}(t_r^{(1)})$,

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[5]More precisely, it is the subset in this product, consisting of sequences of Laurent series in $q - 1$ with all but finitely many terms having no pole when considered modulo any finite term of the filtration in Λ.

and for $\zeta \neq 1$, each $\mathcal{L}_r^{(\zeta)} \subset (\mathcal{K}_r^{(\zeta)})_+ \oplus (\mathcal{K}_r^{(\zeta^{-1})})_-$ is the graph of the self-adjoint linear map $t_r^{(\zeta)} \mapsto d^2 \mathcal{F}_{X/\mathbb{Z}_{m(\zeta)r}}^{tw}(t_{m(\zeta),r}^{(1)}) \, t_r^{(\zeta)}$.

Our reasoning based on summation over trees implies that replacing $\underline{\mathcal{F}}_m$ with $\mathcal{F}_m$ can be considered as the change of the standard polarization into the uniform one without changing the Lagrangian submanifold $\underline{\mathcal{L}}_r \subset \underline{\mathcal{K}}^{(r)}$ which represents the graph of the differential.

Our next goal is to see how all ∇_{mr} intertwine $\mathcal{F}_r$ with $\mathcal{F}_{mr}$ (acting through the parameters $t_{m,r}^{(1)}$ on the former, and through the variables $t_{mr}^{(\zeta)}$ on the latter), and extract terms of weight $\hbar^{-1}$ from the whole expression. We claim that the terms of the total weight $\hbar^{-r}$ are obtained by Wick's summation over *rooted* trees, where the root vertex is represented by $\mathcal{F}_r$, all other vertices correspond to $\mathcal{F}_{mr}$ with $m > 1$, and the edges, connecting vertices of lower *level mr* with higher level *mlr*, come from the propagators ∇_{mlr}.
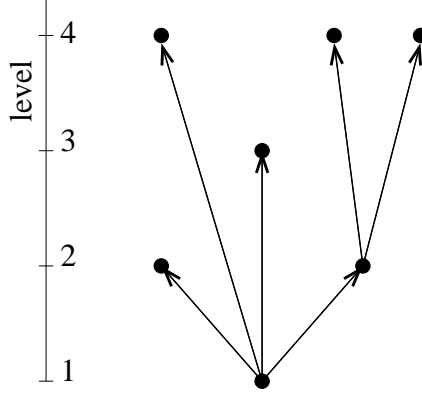

FIGURE 1. A rooted tree

This is a purely combinatorial statement. Orient the edges from lower level to higher (Figure 1). For a tree just described, each non-root vertex of level mr comes with the weight $\hbar^{-mr}$ and a unique *entering* edge weighted by the inverse factor $\hbar^{mr}$. Therefore the whole rooted tree contributes with the weight $\hbar^{-r}$ of the root vertex. Conversely, suppose that a tree has $k + 1 > 1$ "roots" (i.e. vertices with no entering edges). Then there are k edges in excess of those entering non-root vertices and canceling their weights as above. Moreover, since the part of the tree below any fixed level must remain cycle-free, the weights $-n_i$, $n_0 \leqslant \cdots \leqslant n_k$ of the roots are majorated by the weights $m_1 \leqslant \cdots \leqslant m_k$ of the excess edges this way: $n_i < m_i$. In effect, due to the divisibility properties of the weights, $2n_i \leqslant m_i$. Therefore

$n_0 + \cdots + n_k \leqslant m_1 + \cdots + m_k$, and hence the total weight of the tree is non-negative. Finally, adding cycle-generating edges to a tree (even with one root) makes the weight of the resulting graph non-negative.

Let us examine now the summation over trees with one root of level $r = 1$. It affects the root contribution $\mathcal{F}_1$ by shifting the values of the parameters $t = (t_{1,1}^{(1)}, \ldots, t_{m,1}^{(1)}, \ldots)$. This follows from Taylor's formula

$$\mathcal{F}_1(t + y) = \sum_{\alpha = (\alpha_1, \ldots, \alpha_m, \ldots)} \partial^\alpha \mathcal{F}_1(t) \frac{y^\alpha}{\alpha!},$$

where y_m signifies the component of the derivation $\nabla_m \mathcal{F}_m$ (recall that we view ∇_m as a bi-derivation) in the direction of the variables $t_{m,1}^{(1)}$ (and dropping the components in the directions of $t_m^{(\eta)}$ with $\eta \neq 1$). Of course, the variables $t_m^{(\zeta)}$ of $\mathcal{F}_m$ together with partial derivatives of $\mathcal{F}_m$ in these variables (computed at certain values of the parameters $t_{l,m}^{(1)}$) form a point on the graph of the differential of $\mathcal{F}_m$ (at these values of the parameters), which is a Lagrangian submanifold in the symplectic loop space $\underline{\mathcal{K}}^{(m)} := \prod_\zeta \mathcal{K}_m^{(\zeta)}$. Let us denote this point by $\underline{f}_m$, and the Lagrangian submanifold by $\underline{\mathcal{L}}_m$. Since ∇_m generates the transition between the standard and uniform polarizations on $\underline{\mathcal{K}}^{(m)}$, we conclude that the shifted value $t_m^{(1)} + y_m$ of the parameter $t_{m,1}^{(1)} = t_m^{(1)}$ can be described as the component in the untwisted sector of the projection $[\underline{f}_m]_+$ of $\underline{f}_m \in \underline{\mathcal{L}}_m \subset \underline{\mathcal{K}}^{(m)}$ to $\underline{\mathcal{K}}_+^{(m)}$ with respect to the *uniform* polarization (while with respect to the standard one, such projection is t_m by the very definition).

The same is true for the terms weighted by $\hbar^{-r}$, which come from rooted trees with the root at the level r: the effect of all propagators ∇_{mlr} on $\mathcal{F}_r$ consists in shifting the values of the parameters $t_{rm}^{(1)}$ into $[\underline{f}_{rm}]_+^{(1)}$ where $\underline{f}_{rm} \in \underline{\mathcal{L}}_{rm} \subset \underline{\mathcal{K}}^{(rm)}$ represents the differential $d\mathcal{F}_{rm}$ at the point $\{t_{rm}^{(\eta)}\}$ and at the appropriate values of the parameters $t_{rlm}^{(1)}$. The subscript in $[\ldots]_+^{(1)}$ refers to the projections along the negative space of the *uniform* polarization in $\underline{\mathcal{K}}^{(rm)}$, while the superscript indicates the component in the untwisted sector $\eta = 1$. Note that the "appropriate" values of the parameters $t_{rlm}^{(1)}$ are likewise determined by the differentials of $\mathcal{F}_n$, where n runs multiples of rlm, so that this description has the form of an infinite recursion relation.

Our next goal is to identify $\mathcal{L}_r^{(\zeta)}$ in terms of the genus-0 descendant potential $\mathcal{F}_X^{fake}$ of the fake (non-twisted) quantum K-theory of X. We denote by $\mathcal{L}_X^{fake} \subset \mathcal{K}^{fake}$ the overruled Lagrangian cone (see [1, 3, 5])

representing the graph of the differential of $\mathcal{F}_X^{fake}$ with respect to the Lagrangian polarization $\mathcal{K}_{\pm}^{fake}$ on $(\mathcal{K}^{fake}, \Omega^{fake})$ (and dilaton-shifted by $(1-q)\mathbf{1}$). In the symplectic space $\mathcal{K}_r^{(1)}$ (it is equipped with the symplectic form based on the twisted pairing $(\cdot, \cdot)^{(r)}$, lies the Lagrangian cone $\Psi^r(\mathcal{L}_X^{fake})$. In fact, this is $\mathcal{L}_r^{(1)}$.

To prove it, let us express in terms of $\mathcal{F}_X^{fake}$ the function $\mathcal{F}_{X/\mathbb{Z}_r}^{tw}$ at the unramified input $t_r^{(1)} = t$ and with all inputs from ramified sectors set to zero. We claim that[6]

$$\mathcal{F}_{X/\mathbb{Z}_r}^{tw}(t, Q^r) = r^{-1}\Psi^r(\mathcal{F}_X^{fake}(\Psi^{1/r}t, Q)).$$

Indeed, let $\mathcal{M}$ temporarily denote a moduli space, $X_{0,n,d}$, of genus-0 stable maps to X, so that its contribution to $\mathcal{F}_X^{fake}(t)$ is given by

$$\chi^{fake}(\mathcal{M}; V) = \int_{[\mathcal{M}]} \mathrm{ch}(V)\,\mathrm{td}(T_{\mathcal{M}}),$$

where $V = \otimes_{i=1}^n \left(\sum_m \mathrm{ev}_i^*(t_m)L_i^m\right)$, and $T_{\mathcal{M}}$ is the virtual tangent bundle to $\mathcal{M}$. The same moduli space parameterizes stable maps to $X/\mathbb{Z}_r$, which are *trivial* $\mathbb{Z}_r$-covers of stable maps to X. Its contribution to $\mathcal{F}_{X/\mathbb{Z}_r}$ (non-twisted) equals $r^{-1}\chi^{fake}(\mathcal{M}; V)$, where the factor r^{-1} comes from $\mathbb{Z}_r$-symmetries of the covers. The contribution into $\mathcal{F}_{X/\mathbb{Z}_r}^{tw}$ is different, and is equal to

$$r^{-1}\chi^{fake}\left(\mathcal{M}; V \otimes \frac{\mathrm{Eu}(T_{\mathcal{M}})}{\mathrm{Eu}(\Psi^r(T_{\mathcal{M}}))}\right).$$

The twisting fraction $\mathrm{Eu}(T_{\mathcal{M}})/\mathrm{Eu}(\Psi^r(T_{\mathcal{M}}))$ reflects the difference between deformations of a trivial $\mathbb{Z}_r$-cover as stable maps to $X/\mathbb{Z}_r$ and as a stable map to X of a disconnected curve with r components. For each K-theoretic Chern root L of the virtual tangent bundle, we have (putting $x = c_1(L)$):

$$\mathrm{td}(L)\left(\mathrm{ch}\,\frac{Eu(L)}{\mathrm{Eu}(L^r)}\right) = \frac{x}{1-e^{-x}}\frac{1-e^{-x}}{1-e^{-rx}} = \frac{1}{r}\mathrm{td}(L^r).$$

Therefore in cohomological terms, the contribution into $\mathcal{F}_{X/\mathbb{Z}_r}^{tw}$ reads:

$$r^{-1-\dim_{\mathbb{C}} T_{\mathcal{M}}} \int_{[\mathcal{M}]} \mathrm{ch}(V)\,\mathrm{td}(\Psi^r(T_{\mathcal{M}})).$$

The integrand here can be rewritten as $\Psi^r(\mathrm{ch}(\Psi^{1/r}(V))\,\mathrm{td}(T_{\mathcal{M}}))$. Note that Ψ^r acts on cohomology as multiplication by $r^{degree/2}$. Since integration over the virtual fundamental cycle $[\mathcal{M}]$ picks only the terms of

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[6]Ψ^r is invertible in *fake* K-theory.

degree $2 \dim_{\mathbb{C}} T_{\mathcal{M}}$, the result comes out the same as

$$r^{-1} \int_{[\mathcal{M}]} \mathrm{ch}(\Psi^{1/r}(V)) \, \mathrm{td}(T_{\mathcal{M}}) = r^{-1} \chi^{fake}(\mathcal{M}; \Psi^{1/r}(V)).$$

Taking into account that the polarizations in $\mathcal{K}_1^{(1)} = \mathcal{K}^{fake}$ and $\mathcal{K}_r^{(1)}$ are also related by the operation Ψ^r, we conclude[7] that the graph $\mathcal{L}_r^{(1)} \subset \mathcal{K}_r^{(1)}$ of the differential of $\mathcal{F}_{X/\mathbb{Z}_r}^{tw}$ in the unramified sector (i.e. after setting $t_r^{(\zeta)} = 0$ for all $\zeta \neq 1$) coincides with $\Psi^r(\mathcal{L}_X^{fake})$.

Next, from Section 3 of Part IX, we have (in dilaton-shifted notation)

$$\mathcal{F}_{X/\mathbb{Z}_M}\left(\{\mathbf{t}^{(h)}\}_{h\in\mathbb{Z}_r}\right) = \sum_{\chi\in\mathrm{Repr}(\mathbb{Z}_r)} \mathcal{F}_X^{fake}\left(\frac{1}{M}\sum_{\chi} t^{(h)}\chi(h)\right).$$

The quadratic differential of $\mathcal{F}_{X/\mathbb{Z}_M}$ at an unramified input is

$$\frac{1}{2M}\sum_{h\neq 1}\left\langle d^2\mathcal{F}_X^{fake}(t^{(1)})\, t^{(h)}, t^{(h^{-1})}\right\rangle,$$

where we took into account that dilaton-shifted 2nd derivatives of $\mathcal{F}_X^{fake}$ are homogeneous of degree 0. The Lagrangian subspace generated by the quadratic form $\frac{1}{2}\left\langle d^2\mathcal{F}_X^{fake}(t)u, u\right\rangle$, i.e. the graph of the linear map $u \mapsto d^2\mathcal{F}_X^{fake}(t)u$, is the tangent space T to $\mathcal{L}_X^{fake}$ at the point corresponding to $d\mathcal{F}_X^{fake}$ at the input t. Now the transformation from (non-twisted) $\mathcal{F}_{X/\mathbb{Z}_M}$ to $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$ described in Section 4 of Part IX shows that the graph $\mathcal{L}_{r(\zeta)}^{(\zeta)}$ of the linear map $u^{(\zeta)} \mapsto d^2\mathcal{F}_{X/\mathbb{Z}_M}^{tw}(t)u^{(\zeta)}$ (defined by 2nd derivatives of $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}$ in sectors labeled by $\zeta^{\pm 1}$) is obtained from T by the multiplication operator (where $r = r(\zeta), m = m(\zeta)$)

$$\square_{\zeta,r} = e^{\sum_{k>0}\left(\frac{\Psi^{kr}(T_X^*-1)}{k(1-\zeta^{-k}q^{kr/m})} - \frac{\Psi^k(T_X^*-1)}{k(1-q^k)}\right)}.$$

Note that our previous result $\mathcal{F}_{X/\mathbb{Z}_M}^{tw}(t) = M^{-1}\mathcal{F}_X^{fake}(\Psi^{1/M}(t))$ implies that $\square_{1,M}T = \Psi^M(T)$. Therefore

$$\mathcal{L}_r^{(\zeta)} = \square_{\zeta,r(\zeta)}\square_{1,M}^{-1}\Psi^M(T) = \Psi^r(\triangle_\zeta)\Psi^M(T) = \Psi^r(\triangle_\zeta\Psi^m(T)),$$

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[7]The coefficient r^{-1} is absorbed by the fact the symplectic form in $\underline{\mathcal{K}}^{(r)}$ satisfies $\underline{\Omega}^{(r)}(\Psi^r(f), \Psi^r(g)) = r^{-1}\Psi^r(\Omega^{fake}(f,g))$. Indeed, due to degree-2 homogeneity of $\mathcal{F}_X^{fake}$, for a point $f \in \mathcal{L}_X^{fake}$, we have $\mathcal{F}_X^{\mathrm{fake}}([f]_+) = \Omega^{fake}(f,[f]_+)/2$, and hence $\underline{\Omega}^{(r)}(\Psi^r(f), \Psi^r([f]_+))/2 = r^{-1}\Psi^r(\mathcal{F}_X^{fake}([f]_+))$.

where $m = m(\zeta)$, $r = r(\zeta)$, $M = mr$, and

$$\triangle_\zeta := e^{\sum_{k>0} \left(\frac{\Psi^k(T_X^*-1)}{k(1-\zeta^{-k}q^{k/m})} - \frac{\Psi^{km}(T_X^*-1)}{k(1-q^{km})} \right)}.$$

To be more accurate, one needs to tensor the result with Λ over $\Psi^M(\Lambda)$ since Ψ^M may not be invertible on the coefficient ring.

Finally, let us combine all our previous observations on the adelic data with the description of the adelic map which, according to the main theorem of Part IX, induces $\langle \mathcal{D}_X \rangle$ from $\langle \underline{\mathcal{D}}_X \rangle$. The term $\mathcal{F}_X$ in $\log \mathcal{D}_X$ is induced this way from terms of weight $\hbar^{-1}$ in $\log \underline{D}_X$ *after* the change of standard polarization to the uniform one.

Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_r, \ldots)$ be a point on the graph $\mathcal{L}_X \subset \mathcal{K}^\infty$ of the differential of $\mathcal{F}_X$. Then for each r, $\underline{\mathbf{f}}_r = \{\underline{\mathbf{f}}_r(q^{1/m(\zeta)}/\zeta)\}$ lies in $\underline{\mathcal{L}}_r = \prod_\zeta \mathcal{L}_r^{(\zeta)}$, which is the graph of the differential of $\mathcal{F}_r$ (in fact of one of the functions of the family $\mathcal{F}_r$, taken at an appropriate value of the parameter) with respect to the uniform polarization.

More concretely, decompose each vector-valued rational function $\mathbf{f}_r$ into the sum of a Laurent polynomial $\mathbf{t}_r \in \mathcal{K}_+^{(r)}$ and simple fractions $\mathbf{p}_r^{(\eta)}$ with poles (or any order) at a root of unity $q = \eta$: $\mathbf{f}_r = \mathbf{t}_r + \sum_\eta \mathbf{p}_r^{(\eta)}$. Computing the adelic map, $\underline{\mathbf{f}}_r = \{\underline{\mathbf{f}}_r^{(\zeta)}\}$, for a primitive mth root of unity ζ, we find[8]

$$\underline{\mathbf{f}}_r^{(\zeta)} = \Psi^r(\mathbf{p}_r^{(\zeta)}(q^{1/m}/\zeta)) \quad (\mathcal{K}_-^{(\zeta)}\text{-part in the standard polarization})$$

$$+ \Psi^r \left(\mathbf{t}_r(q^{1/m}/\zeta) + \sum_{\eta \neq \zeta} \mathbf{p}_r^{(\eta)}(q^{1/m}/\zeta) \right) \quad (\mathcal{K}_+^{(\zeta)}\text{-part}).$$

In particular, if we put $\mathbf{t}_r = 0$, these expansions would describe vectors of the negative space in the *uniform* polarization. For a general value of $\mathbf{t}_r$, projections along such vectors yields $\Psi^r(\mathbf{t}_r(q^{1/m}/\zeta))$. This means that $\underline{\mathbf{f}}_r$ represents with respect to the *uniform* polarization the differential $d\mathcal{F}_r$, computed at the input $\{t_r^{(\zeta)}\}$ with $t_r^{(\zeta)} = \Psi^r(\mathbf{t}_r(q^{1/m}/\zeta))$. Equivalently: $\underline{\mathbf{f}}_r$ represents with respect to the *standard* polarization the differential of $d\underline{\mathcal{F}}_r$, computed at the point where $t_r^{(\zeta)} = \Psi^r(\mathbf{t}_r(q^{1/m}/\zeta) + \sum_{\eta \neq \zeta} \mathbf{p}_r^{(\eta)}(q^{1/m}/\zeta))$. Since the graph of $d\underline{\mathcal{F}}_r$ is the product $\underline{\mathcal{L}}_r = \prod_\zeta \mathcal{L}_r^{(\zeta)}$, we conclude that $\underline{\mathbf{f}}_r^{(\zeta)} \in \mathcal{L}_r^{(\zeta)}$.

Now recall that $\mathcal{L}_r^{(1)} = \Psi^r(\mathcal{L}_X^{fake})$ to conclude that $\mathbf{f}_r^{(1)} = \Psi^{1/r}(\underline{\mathbf{f}}_r^{(1)})$ lies in $\mathcal{L}_X^{fake}$. This confirms the criterion (i) in Theorem 1.

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[8]Expansions of rational functions into Laurent series near $q = 1$ are tacitly assumed on the right.

Furthermore, to verify the criterion (ii), take $\zeta \neq 1$, and recall that $\mathcal{L}_r^{(\zeta)} = \Psi^r(\triangle_\zeta \Psi^m(T)) \otimes \Lambda$, where T is some tangent space to $\mathcal{L}_X^{fake}$. We conclude that $\mathbf{f}_r^{(\zeta)} = \Psi^{1/r} \underline{\mathbf{f}}_r^{(\zeta)}$ lies in $\triangle_\zeta \Psi^m(T) \otimes \Lambda$.

How is the space T determined? It is $T_f \mathcal{L}_X^{fake}$, where $[\Psi^M(f)]_+$ ($M = mr$) is the value of the parameter $t_{m,r}^{(1)}$ in the family of functions $\mathcal{F}_r$. How is that value determined? It is the untwisted component $[\underline{\mathbf{f}}_{mr}]_+^{(1)}$ of the point on the graph of $d\mathcal{F}_{mr}$. As we already know, $\underline{\mathbf{f}}_{mr}^{(1)}$ lies in $\Psi^{mr}(\mathcal{L}_X^{fake})$, and hence $\mathbf{f}_{mr}^{(1)} = \Psi^{1/mr}(\underline{\mathbf{f}}_{mr}^{(1)})$ lies in $\mathcal{L}_X^{fake}$. This shows $f = \mathbf{f}_{mr}^{(1)}$ and therefore $T = T_{\mathbf{f}_{mr}^{(1)}} \mathcal{L}_X^{fake}$ as required.

Ultimately we conclude that if $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots)$ is a point in $\mathcal{L}_X$, then the criteria (i) and (ii) of Theorem 1 must be satisfied:

$$(i) \ \ \mathbf{f}_r^{(1)} \in \mathcal{L}_X^{fake}, \ \ \text{and} \ \ (ii) \ \ \mathbf{f}_r^{(\zeta)} \in \triangle_\zeta \Psi^m(T_{\mathbf{f}_{rm}^{(1)}} \mathcal{L}_X^{fake}) \otimes_{\Psi^m(\Lambda)} \Lambda$$

for every $r = 1, 2, \ldots$, and every primitive mth root of unity $\zeta \neq 1$.

From the computational point of view, this is an infinite system of recursion relations. However, as it is easy to see, modulo any fixed power of the ideal Λ_+ (or, more explicitly, under a limited degree and/or the number of marked points on the curves) this yields a finite system of recursion relations which uniquely determines $\mathbf{f} \in \mathcal{L}_X$ from the projection $\mathbf{t}$ of $\mathbf{f}$ to $\mathcal{K}_+^\infty$.

Thus, the criteria (i) and (ii) completely characterize points in $\mathcal{L}_X$ which in a Λ_+-neighborhood of the dilaton vector.

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