PERMUTATION-EQUIVARIANT QUANTUM K-THEORY I. DEFINITIONS. ELEMENTARY K-THEORY OF $\overline{\mathcal{M}}_{0,n}/S_n$

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Abstract. K-theoretic Gromov-Witten (GW) invariants of a compact Kähler manifold X are defined as super-dimensions of sheaf cohomology of interesting bundles over moduli spaces of n-pointed holomorphic curves in X. In this paper, we introduce K-theoretic GW-invariants cognizant of the S_n-module structure on the sheaf cohomology, induced by renumbering of the marked points, and compute some of these invariants for $X = pt$.

Preface

In Fall 2014, I gave a talk on the subject of permutation-equivariant quantum K-theory and its relations to mirror symmetry at *The Legacy of Vladimir Arnold* conference in Toronto. Explaining afterwards why the work had not been published yet, I received a piece of good advice from Anatoly Vershik, who suggested that one should publish not a whole theory, but small portions of it.

The present paper begins a series of such portions. Each one is supposed to have its own punch-line, and be reasonably self-contained, or at least readable separately from the others. Yet, they are chapters of the same story, follow a single plan, and are meant *to be continued*. One of our intentions is to identify the right place for toric q-hypergeometric functions among genus-0 K-theoretic Gromov–Witten invariants. Another one is to elucidate the role of finite-difference operators. In particular, we will see that the q-exponential function is even more prominent in quantum K-theory then the ordinary exponential function is in quantum cohomology. As a remote goal, we would like the q-analogues of the Witten–Kontsevich tau-function to arise from K-theory of the Deligne–Mumford quotients $\overline{\mathcal{M}}_{g,n}/S_n$.

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S_n-EQUIVARIANT CORRELATORS

Let X be a compact Kähler manifold, a *target space* of GW-theory, $X_{g,n,d}$ denote the moduli space of degree-d stable maps to X of nodal compact connected n-pointed curves of arithmetical genus g, ev_i : $X_{g,n,d} \to X$ the evaluation map at the ith marked point, L_i the line bundle over $X_{g,n,d}$ formed by the cotangent lines to the curves at the ith marked point. Given elements $\phi_i \in K^0(X)$ and integers $k_i \in \mathbb{Z}$, $i = 1, \dots, n$, one defines a K-theoretic GW-invariant of X as the holomorphic Euler characteristic

$$\langle \phi_1 L^{k_1}, \dots, \phi_n L^{k_n} \rangle_{g,n,d} := \chi\left(X_{g,n,d}; \mathcal{O}^{virt}_{g,n,d} \otimes \prod_{i=1}^n L_i^{k_i}\, \mathrm{ev}_i^*(\phi_i)\right).$$

Here $\mathcal{O}^{virt}_{g,n,d}$ is the *virtual structure sheaf* introduced by Y.-P. Lee [3] as the K-theoretic counterpart of virtual fundamental cycles in the cohomological theory of GW-invariants. The above "correlators" can be extended poly-linearly to the space of Laurent polynomials

$$t(q) = \sum_{m\in\mathbb{Z}} t_m q^m, \quad t_m = \sum_\alpha t_{m,\alpha}\phi_\alpha$$

(here $\{\phi_\alpha\}$ is a basis in $K^0(X)\otimes\mathbb{Q}$, and $t_{k,\alpha}$ are formal variables), and thereby encode the values of all individual correlators by the totally symmetric degree-n polynomial $\langle t(L), \dots, t(L)\rangle_{g,n,d}$.

Our aim is to enrich this information using the action of S_n by permutations of the marked points. Namely, since the marked points are numbered, their renumbering on a given stable map produces a new stable map, and hence this operation induces an automorphism of the moduli space: $X_{g,n,d} \to X_{g,n,d}$. In fact the automorphism is relative over $X_{g,0,d}$ (here we have in mind the map $\mathrm{ft} : X_{g,n,d} \to X_{g,0,d}$ defined by forgetting the marked point). The map ft respects the construction [3] of virtual structure sheaves:

$$\mathcal{O}^{virt}_{g,n,d} = \mathrm{ft}^*\, \mathcal{O}^{virt}_{g,0,d}.$$

Therefore, as long as the *inputs* $t(q)$ in all seats of the correlator are the same, the corresponding sheaf cohomology, and hence their alternated sum, carries a well-defined structure of a virtual S_n-module. Let us introduce for this S_n-module the notation

$$[t(L),\dots,t(L)]_{g,n,d} := \sum(-1)^m H^m(X_{g,n,d};\mathcal{O}^{virt}_{g,n,d}\otimes\prod_{i=1}^n(\sum_{k\in\mathbb{Z}}\mathrm{ev}_i^*(t_k)L_i^k)).$$

Thus defined GW-invariants with values in the representation ring of S_n lack two features required by the standard combinatorial framework of GW-theory: they are not poly-linear, and they take incomparable values for different values of n. We handle both difficulties by employing Schur–Weyl reciprocity.

Let Λ be a λ-algebra, by which we will understand an algebra over $\mathbb{Q}$ equipped with abstract *Adams operations* Ψ^m, $m = 1,2,\dots$, i.e. ring homoorphisms $\Lambda\to\Lambda$ satisfying $\Psi^r\Psi^s = \Psi^{rs}$ and $\Psi^1 = \mathrm{id}$. [1] The following construction of correlators has direct topological meaning when $\Lambda = K^0(Y)\otimes\mathbb{Q}$, the K-ring of some space Y equipped with the natural Adams operations, but it can be extended to arbitrary λ-algebras.

On the role of *inputs* we take Laurent polynomials $\mathbf{t} = \sum_{m\in\mathbb{Z}}\mathbf{t}_m q^m$ with vector coefficients $\mathbf{t}_k\in K^0(X)\otimes\Lambda$. Given several such inputs $\mathbf{t}^{(1)},\dots\mathbf{t}^{(s)}$, we define correlators of *permutation-equivariant* quantum K-theory with several groups of sizes $k_1+\dots+k_s = n$ of identical inputs (and hence symmetric with respect to the subgroup $H = S_{k_1}\times\dots\times S_{k_s}$ of S_n), and taking values in Λ:

$$\langle\mathbf{t}^{(1)},\dots,\mathbf{t}^{(1)};\dots;\mathbf{t}^{(s)},\dots,\mathbf{t}^{(s)}\rangle^H_{g,n,d} :=$$
$$(\pi : (X_{g,n,d}\times Y)/H\to Y)_*\left(\mathcal{O}^{virt}_{g,n,d}\otimes\prod_{a=1}^s\prod_{i=1}^{k_a}\left(\sum_{m\in\mathbb{Z}}\mathrm{ev}_i^*(\mathbf{t}^{(a)}_m)L_i^m\right)\right)$$

where π_* is the K-theoretic push-forward along the indicated projection map π. Note that the sheaf on the right lives naturally on $X_{g,n,d}\times Y$ and is H-invariant (where the action on Y is meant to be trivial). Taking the quotient, by definition, extracts H-invariants from the K-theoretic push-forward to Y.

`Example 1.` *GL_N-equivariant K-theory.* Take Λ to be the algebra of symmetric functions in N variables $x_1,\dots,x_N$ with the Adams operations $\Psi^r(x_i) = x_i^r$. It can be viewed as (a subring in) $\mathit{Repr}\,GL_N = K^0(BGL_N(\mathbb{C}))$, the representation ring of $GL_N(\mathbb{C})$, by considering x_i

[1] One usually defines λ-algebras in terms of axiomatic exterior power operation. For us the Adams operations will be more important. The difference disappears over $\mathbb{Q}$. The reason is that Newton polynomials are expressed as polynomials with integer coefficients in terms of elementary symmetric functions, but the inverse formulas involve fractions.

to be the eigenvalues of diagonal matrices in the vector representation $\mathbb{C}^N$. Respectively, $K^0(X)\otimes\Lambda$ can be interpreted as GL_N-equivariant K-ring of X equipped with the trivial GL_N-action. Let t be a legitimate input of the ordinary quantum K-theory, i.e. Laurent polynomial L with coefficients from $K^0(X)$, and $\nu\in\Lambda$. Then

$$\langle \nu t,\dots,\nu t\rangle_{g,n,d}^{S_n}=\frac{1}{n!}\sum_{h\in S_n}\mathrm{tr}_h[t,\dots,t]_{g,n,d}\prod_{r=1}^{\infty}\Psi^r(\nu)^{l_r(h)},$$

where $l_r(h)$ denotes the number of cycles of length r in the permutation h. Indeed, if ν in the correlator stands for a GL_N-module attached at each marked point, then $[\nu t,\dots,\nu t]_{g,n,d}=[t,\dots t]_{g,n,d}\otimes\nu^{\otimes n}$. The second factor here is a $GL_N(\mathbb{C})\times S_n$-module. For a diagonal matrix x and a permutation h, we have

$$\mathrm{tr}_{(x,h)}\,\nu^{\otimes n}=\mathrm{tr}_x\prod_{r=1}^{\infty}\Psi^r(\nu)^{l_r(h)}.$$

Indeed, due to the universality of Adams operations, it suffices to check this for $\nu=\mathbb{C}^N$, the vector representation, which is straightforward:

$$\mathrm{tr}_{(x,h)}(\mathbb{C}^N)^{\otimes n}=\prod_{r=1}^{\infty}\nu_r^{l_r(h)}(x),$$

where $\nu_r(x)=x_1^r+\dots+x_N^r=\Psi^r(\nu_1)$ is the rth Newton polynomial.

`Example 2:` *Schur–Weyl's reciprocity.* According to Schur–Weyl's reciprocity, the $GL_N\times S_n$-character of $(\mathbb{C}^N)^{\otimes n}$ has the form:

$$\prod_{r=1}^{\infty}\nu_r^{l_r(h)}(x)=\sum_{\triangle}w_\triangle(h)s_\triangle(x),$$

where $s_\triangle$ is the Schur polynomial, the character of the irreducible GL_N-module with the highest weight determined by the partition (or the Young diagram) $\triangle$, and $w_\triangle$ is the character of the irreducible S_n-module corresponding to the same Young diagram. The diagrams here consist of n cells and have no more than N rows. The Schur polynomials $s_\triangle$ form a real orthonormal basis in the space of all symmetric polynomials of degree n (in N variables). Therefore, using the notation $(\cdot,\cdot)$ for pairing of representations (or characters), we have:

$$(\langle t(L),\dots,t(L)\rangle_{g,n,d}^{S_n},s_\nabla)=\frac{1}{n!}\sum_{h\in S_n}\mathrm{tr}_h[t(L),\dots,t(L)]_{g,n,d}w_\nabla(h),$$

that is, equal to the multiplicity of the irreducible S_n-module ∇ in the S_n-module of our interest.

`Example 3:` $N \to \infty$. In this limit, Λ becomes the abstract algebra of symmetric functions $\mathbb{Q}[[\nu_1, \nu_2, \dots]]$ with the Adams operations $\Psi^r(\nu_m) = \nu_{rm}$. This example captures the entire information about $[t, \dots, t]_{g,n,d}$ as S_n-modules for all n simultaneously.

`Example 4:` *Symmetrized quantum K-theory.* Taking in Example 1 $N = 1$, we obtain $\Lambda = \mathbb{Q}[x]$ with $\Psi^r(x) = x^r$. This choice corresponds extracting S_n invariants from sheaf cohomology:

$$[xt, \dots, xt]_{g,n,d} = [t, \dots, t]^{S_n}_{g,n,d} x^n.$$

Indeed, the action of S_n on GL_1-module $(\mathbb{C}^1)^{\otimes n}$ is trivial. We will refer to this important special case of permutation-equivariant quantum K-theory as *permutation-invariant* or *symmetrized.*

`Example 5:` *The permutation-equivariant binomial formula.* Returning to the definition of permutation-equivariant correlators, we can see that they possess permutation-equivariant version of poly-additivity. For instance,

$$\langle \mathbf{t}' + \mathbf{t}'', \dots, \mathbf{t}' + \mathbf{t}'' \rangle^{S_n}_{g,n,d} = \sum_{k+l=n} \langle \mathbf{t}', \dots \mathbf{t}', \mathbf{t}'', \dots \mathbf{t}'' \rangle^{S_k \times S_l}_{g,n,d}.$$

Using the bracket notation $\langle \dots \rangle$ for the sheaf cohomology on $X_{g,n,d} \times Y$ (i.e. before taking S_n-invariants), we have the following equality of S_n-modules:

$$\langle \mathbf{t}' + \mathbf{t}'', \dots, \mathbf{t}' + \mathbf{t}'' \rangle_{g,n,d} = \sum_{k+l=n} \operatorname{Ind}^{S_n}_{S_k \times S_l} \langle \mathbf{t}', \dots, \mathbf{t}', \mathbf{t}'', \dots, \mathbf{t}'' \rangle_{g,n,d},$$

where Ind^G_H denotes the operation of inducing a G-module from an H-module. Extracting S_n-invariants on both sides proves the claim. Indeed, due to the reciprocity between inducing and restricting, for any H-module V, we have $(\operatorname{Ind}^G_H V)^G = V^H$, since restricting the trivial G-module to H yields the trivial H-module.

Finally introduce the genus-g *descendent potentials* of permutation-equivariant quantum K-theory:

$$\mathcal{F}_g = \sum_{d,n} Q^d \langle \mathbf{t}(L), \dots, \mathbf{t}(L) \rangle^{S_n}_{g,n,d}.$$

Here Q^d is, as usual, the monomial representing the degree $d \in H_2(X)$ in the Novikov ring. Note that the customary in Taylor's formulas division by $n!$ is replaced by extracting S_n-invariants. The potential is a formal function on the space of Laurent polynomials in q with coefficients in $K^0(X) \otimes \Lambda$. We assume that λ-algebra Λ is extended to power series in Novikov's variables (e.g. one could take $\Lambda = \mathbb{Q}[[\nu_1, \nu_2, \dots]]\,[[Q]]$)

and the Adams operations are extended by $\Psi^r(Q^d) = Q^{rd}$. We will refer to Λ as *Newton-Novikov's ring*.

`Remark.` I am thankful to A. Polishchuk, who pointed out to me that in a related context of *modular operads*, an equivalent formalism of encoding permutation-equivarint information using the algebra of symmetric functions [4] was used by E. Getzler and M. Kapranov [1].

The small J-function of the point

In this section, we use an explicit description of Deligne–Mumford spaces $\overline{\mathcal{M}}_{0,n}$ in terms of Veronese curves to compute the "small" J-function in the permutation-equivariant quantum K-theory of $X = pt$.

`Theorem.` *For $\nu \in \Lambda$, put*

$$J_{pt}(\nu) := 1 - q + \nu + \sum_{n\geq 2} \langle \nu, \dots, \nu, \frac{1}{1-qL} \rangle_{0,n+1}^{S_n}.$$

Then

$$J_{pt} = (1-q) e^{\sum_{k>0} \Psi^k(\nu)/k(1-q^k)}.$$

`Proof.` We refer to the paper [2] by M. Kapranov for details of the description of $\overline{\mathcal{M}}_{0,n+1}$ in terms of Veronese curves in $\mathbb{C}P^{n-2}$, i.e. generic rational curves of degree equal to the dimension of the ambient projective space. They are all isomorphic to the model Veronese curve $(u : v) \mapsto (u^{n-2} : u^{n-3}v : \dots : uv^{n-3} : v^{n-2})$ under the action of $PGL_2(\mathbb{C}) \times PGL_{n-1}(\mathbb{C})$ by reparameterizations and projective automorphisms, and form a family of dimension $(n+1)(n-3)$. The moduli space $\overline{\mathcal{M}}_{0,n+1}$ is identified with a suitable closure of the space of Veronese curves passing through a fixed generic configuration of n points $p_1, \dots, p_n \in \mathbb{C}P^{n-2}$. According to [2], the closure can be taken in the Chow scheme of algebraic cycles (or in the suitable Hilbert scheme). Moreover, $\overline{\mathcal{M}}_{0,n+1}$ is obtained explicitly by a certain succession of blow-ups of $\mathbb{C}P^{n-2}$ centered at all subspaces passing through the n points. The rational map, inverse to the projection $\pi : \overline{\mathcal{M}}_{0,n+1} \to \mathbb{C}P^{n-2}$, can be described this way: for a generic $p \in \mathbb{C}P^{n-2}$, there is a unique Veronese curve passing through $(p_1, \dots, p_n, p)$. (Example: a unique conic through 5 generic points on the plane.)

The forgetful map $\mathrm{ft}_{n+2} : \overline{\mathcal{M}}_{0,n+2} \to \overline{\mathcal{M}}_{0,n+1}$ can be described as follows (see Figure 1). Veronese curves of degree $n-1$ in $\mathbb{C}P^{n-1}$ passing through fixed generic points $p_1, \dots, p_n, p_{n+1}$ can be projected from p_{n+1} to $\mathbb{C}P^{n-1}$, to become Veronese curves of degree $n-2$ passing through the projections $\widetilde{p}_1, \dots, \widetilde{p}_n$ of $p_1, \dots, p_n$. According to [2], this projection survives the passage to the Chow closure.

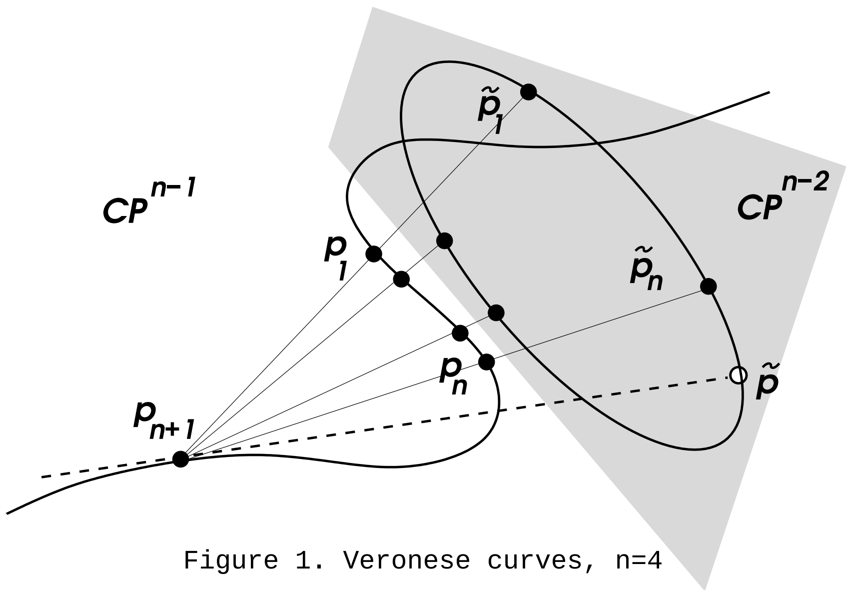

Figure 1. Veronese curves, n=4

Moreover, as it follows from the exact description of the succession of the blow-ups (see [2], Theorem 4.3.3), the section $\overline{\mathcal{M}}_{0,n+1} \subset \overline{\mathcal{M}}_{0,n+2}$ of the forgetful map $\mathrm{ft}_{n+2} : \overline{\mathcal{M}}_{0,n+2} \to \overline{\mathcal{M}}_{0,n+1}$ defined by the $n+1$-st marked point is obtained by blowing up $\mathbb{C}P^{n-1}$ at p_{n+1}, and then taking the proper transform of the exceptional divisor $\mathbb{C}P^{n-2}$ under all further blow-ups. Their centers come from higher-dimensional subspaces passing through p_{n+1}, and are transverse to the the divisor. This means that conormal bundle to the section (which is the official definition of L_{n+1} over $\overline{\mathcal{M}}_{0,n+1}$) coincides with the pull-back of the conormal bundle to the exceptional $\mathbb{C}P^{n-2}$ (which is $\mathcal{O}(1)$) by the blow-down map $\pi : \overline{\mathcal{M}}_{0,n+1} \to \mathbb{C}P^{n-2}$. Thus, $L_{n+1} = \pi^*\mathcal{O}(1)$.

Note that since $L_{n+1} = \pi^*\mathcal{O}(1)$, then $\pi_* L_{n+1}^m = \mathcal{O}(m)$, because the K-theoretic push-forward of the structure sheaf along a blow-down map has trivial higher direct images. Thus the problem of computing J_{pt} receives the following elementary interpretation. Let S_n act on $\mathbb{C}P^{n-2} = \mathrm{proj}(\mathbb{C}^{n-1})$ by permutations of the vertices $p_1, \dots, p_n$ of the standard simplex. Then the S_n-module denoted in the previous section $[1, \dots, 1, L^m]_{0,n+1}$ is the space of degree m polynomials in $\mathbb{C}^{n-1}$. Respectively,

$$\langle \nu, \dots \nu, \frac{1}{1-qL} \rangle_{0,n+1}^{S_n} = \frac{1}{n!} \sum_{h \in S_n} \mathrm{tr}_h \, S_q^*(\mathbb{C}^{n-1}) \prod_{r>0} \Psi^r(\nu)^{l_k(h)},$$

where $S_q^*(\mathbb{C}^{n-1}) = \oplus_{m\geq 0} q^m S^m(\mathbb{C}^{n-1})^*$ is the graded (and weighted by powers of q) algebra of polynomial functions on $\mathbb{C}^{n-1}$.

The series J_{pt}, the total sum of the correlators over all n, can be computed by Lefschetz fixed point formula. In fact summation over all symmetric groups can be rewritten in terms of conjugacy classes. The action of $h \in S_n$ on $\mathbb{C}^n$ (rather than $\mathbb{C}^{n-1}$) decomposes into the direct product of elementary k-cycles c_k acting on $\mathbb{C}^k$ by the cyclic permutation of the coordinates. The trace $\mathrm{tr}_{c_k} S_q^*(\mathbb{C}^k)$ can be computed as $\prod_{s=1}^k (1-e^{2\pi i s/k}q)^{-1} = (1-q^k)^{-1}$, since $e^{2\pi i s}$ are simple eigenvalues of c_k on $\mathbb{C}^k$. Taking in account the size $n!/\prod_k l_k! k^{l_k}$ of the conjugacy class with l_k cycles of length k, we conclude that

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{h\in S_n} \mathrm{tr}_h\, S_q^*(\mathbb{C}^n) \prod_{k>0} \Psi^k(\nu)^{l_k(h)} = \sum_{l_1,l_2,\cdots>0} \prod_{k>0} \frac{1}{l_k!} \left(\frac{\Psi^k(\nu)}{k(1-q^k)}\right)^{l_k}.$$

The latter sum coincides with $e^{\sum_{k>0} \Psi^k(\nu)/k(1-q^k)}$. The extra factor $(1-q)$ in the theorem takes care of the excess (comparing to $\mathbb{C}^{n-1}$) 1-dimensional subspace in $\mathbb{C}^n$ with the trivial action of S^n, because the Poincaré polynomial $\mathrm{tr}_{id}\, S_q^*(\mathbb{C}) = 1/(1-q)$. □

Corollary 1. *In the symmetrized theory, the value of the J-function*

$$J_{pt}^{sym} := 1 - q + x + \sum_{n\geq 2} x^n \dim[\frac{1}{1-qL}, 1, \dots, 1]_{0,n+1}^{S_n}$$

is expressed in terms of the q-exponential function $e_q(y) := \sum_{n\geq 0} \frac{y^n}{[n]_q!}$*:*

$$J_{pt}^{sym} = (1-q) e_q\left(\frac{x}{1-q}\right) = \sum_{n\geq 0} \frac{x^n}{(1-q^2)\dots(1-q^n)}.$$

Proof. Taking in the theorem $\Lambda = \mathbb{Q}[[x]]$ (i.e. choosing GL_N to be GL_1), and sertting $\nu = x$, we find

$$f(x) := (1-q)^{-1} J_{pt}^{sym} = e^{\sum_{k>0} x^k/k(1-q^k)}.$$

Note that f satisfies the following finite-difference equation:

$$f(x) - f(qx) = f(x)\left(1 - e^{-\sum_{k>0} x^k/k}\right) = f(x)(1-(1-x)) = xf(x).$$

For e_q, we also have:

$$e_q(\frac{x}{1-q}) - e_q(\frac{qx}{1-q}) = \sum_{n\geq 0} \frac{x^n(1-q^n)}{(1-q)(1-q^2)\dots(1-q^n)} = xe_q(\frac{x}{1-q}).$$

Since both are power series in x with the free term 1, they coincide. □

`Corollary 2.` *When Λ is the algebra of symmetric functions in $x_1, \ldots, x_N$, and $\nu = t\nu_1$, where t is a scalar, we have*

$$J_{pt}(t\nu_1) = (1-q)\prod_i e_q\left(\frac{x_i}{1-q}\right)^t.$$

`Proof.` Write $\Psi^k(t\nu_1) = t(x_1^k + \cdots + x_N^k)$ for each k. $\square$

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